MAC-CPTM Situations Project

Situation 40: Powers

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Prompt

During an Algebra I lesson on exponents, the teacher asked the students to calculate positive integer powers of 2. A student asked the teacher, "We've found 2^2 and 2^3. What about $2^{2.5}$?"

Commentary

The prompt centers on the extension of the domain of the exponent to numbers beyond integers. The foci explore the nature of exponents numerically, graphically, and analytically. The table with integral values in Focus 1 suggests a pattern for a curve and an extended domain that is illustrated in the graphical representation in Focus 2. Although Foci 1 and 2 provide an estimate, the analytical treatment in Focus 3 generates an exact value. These foci help expand the concept of exponentiation beyond repeated multiplication to accommodate the use of some non-integer exponents.

Mathematical Foci

Mathematical Focus 1

One method for estimating the value of 2^x where $x \in \mathbb{Q}$ uses linear interpolation and the properties of the function f with rule $f(n)=2^n$, where $n \in \mathbb{Z}$.

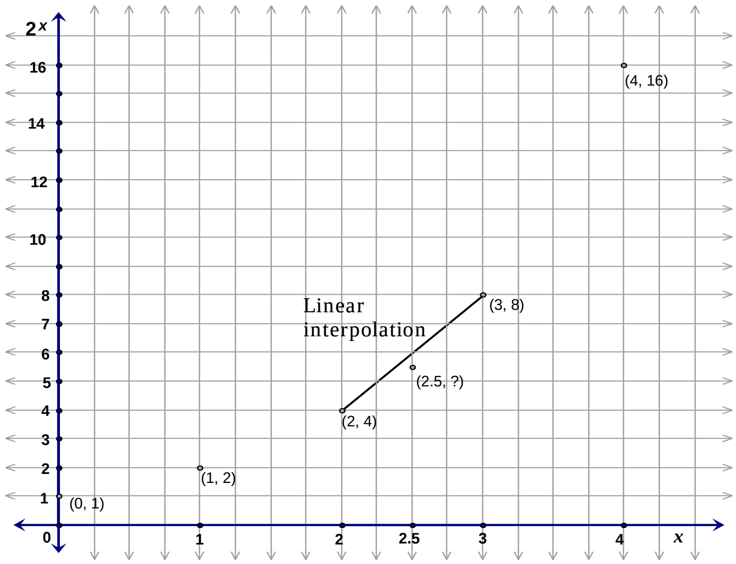
One way to calculate an estimate of the value $2^{2.5}$ is to use linear interpolation between the two known values, 2^2 and 2^3. To use linear interpolation, we treat the function as if it were linear between the known function values. Using linear interpolation, $2^{2.5}$ is approximately 6 (since 2.5 is halfway from 2 to 3, we estimate $2^{2.5}$ to be halfway from 2^2 to 2^3). This is just an estimation however, and it is important to understand that the value for $2^{2.5}$ will not be exactly halfway between 2^2 and 2^3. Whether the value of $2^{2.5}$ is greater than 6 or less than 6 is determined by the pattern of growth of the function. A linear function has constant growth, but we will see that 2^x does not.

The table below shows a pattern of increasing growth between successive values of 2^x and illustrates that exponential growth is different than constant growth. In particular, nonlinearity implies that even though 2.5 is the arithmetic mean of 2 and 3, $2^{2.5}$ will not be the arithmetic mean of 4 and 8.

x	2^x
0	1
1	2
2	4
3	8
4	16

Since the differences between the successive values of 2^x are increasing, one can argue that linear approximations will overestimate the value of $2^{2.5}$. For example, 2^2 is 4, and 2^4 is 16, but 2^3 is less than 10, the value suggested by linear interpolation. Therefore, $2^{2.5}$ should be less than 6.

(Continued)



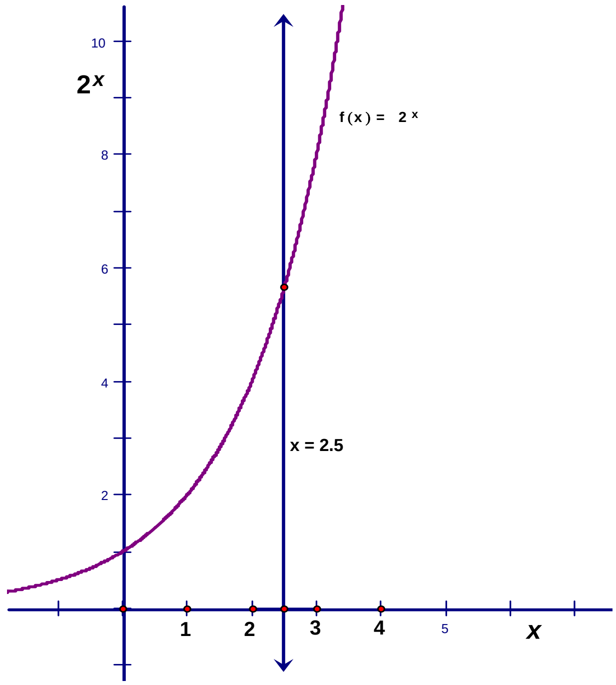
Graphically, we can begin to think of a way to connect the points that will preserve the rate of growth in the table of values. This pattern of points suggests a graph that is concave up. A graph that is concave up increases or decreases at an increasing rate, whereas the graph of a line increases or decreases at a constant rate. We can use this fact to conclude that the value of $2^{2.5}$ will be closer to 4 than to 8.

Mathematical Focus 2

A more accurate estimation of 2^x where $x \notin \mathbb{Z}$ can be obtained through graphical analysis of the function f with rule $f(x) = 2^x,\ x \in \mathbb{R}$.

It should be recognized at this point that one is assuming that the domain of the function *f* with rule $f(x) = 2^x$ can be extended from the set of integers to the set of real numbers. In particular, the resulting graph of the function with the new domain will be represented by a continuous curve. This graph allows one to obtain an estimate for f(2.5) with varying degrees of accuracy depending on the technology or method employed.

For example, one can estimate the value of f(2.5) from a calculator-generated graph of *f* and the trace option. Alternatively, one can look at the intersection of the function graph with the vertical line $x = 2.5$.



Mathematical Focus 3

By identifying rational numbers as ratios of integers, properties of integral exponents are extended to rational exponents.

One possible definition for $b^{\frac{m}{n}}$ is that it is the number that when raised to the n^{th} power gives b^m as a result. So, $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ for $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and $b \in \mathbb{R}^+$. Using this definition, $\left(b^m\right)^{\frac{1}{n}} = \sqrt[n]{\left(b^m\right)^1} = \sqrt[n]{b^m} = b^{\frac{m}{n}}$. We can also establish other properties of rational exponents, such as the following (for m, n, and b as above and $c \in \mathbb{Z}$, $d \in \mathbb{N}$, and $a \in \mathbb{R}^+$):

1. $\left(b^{\frac{1}{n}}\right)^m = \left(b^m\right)^{\frac{1}{n}} = b^{\frac{m}{n}}$
2. $\left(b^{\frac{m}{n}}\right)^{\frac{c}{d}} = \left(b^{\frac{c}{d}}\right)^{\frac{m}{n}} = b^{\frac{mc}{nd}}$
3. $b^{\frac{m}{n}} \cdot b^{\frac{c}{d}} = b^{\frac{c}{d}} \cdot b^{\frac{m}{n}} = b^{\frac{m}{n}+\frac{c}{d}}$
4. $\left(ab\right)^{\frac{m}{n}} = a^{\frac{m}{n}} \cdot b^{\frac{m}{n}}$

Using the representation $2^{\frac{1}{2}} = \sqrt{2}$, we can analyze $2^{2.5} = 2^{\frac{5}{2}}$ as follows: $2^{\frac{5}{2}} = \left(2^5\right)^{\frac{1}{2}} = \sqrt{2^5} = \sqrt{32} \approx 5.657$ or $2^{\frac{5}{2}} = (2^{\frac{1}{2}})^5 = (\sqrt{2})^5 \approx 1.414^5 \approx 5.657$. In another form: $(2^{\frac{1}{2}})^5 = (\sqrt{2})^5 = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = 4\sqrt{2} \approx 5.657$.

Thus, we can define 2^x, with $x \notin \mathbb{Z}$, in a way that allows the familiar properties of exponents to hold. Once we know the properties hold, we can then use them to express $2^{2.5}$ in ways that are, perhaps, easier to understand.

Post – Commentary

It is also useful to consider the fact that $f(x) = b^x$ would behave differently for different values of *b*. If $b < 0$, the properties would require further modification. Furthermore, different mathematical discussions would be necessary if $b \notin \mathbb{R}$.

Although it is not strictly necessary, it is common to place the above restrictions on the values of *m* and *n* as well, depending on the restrictions placed on *b*. For example, $3^{\frac{1}{-4}} = \sqrt[-4]{3}$ is not considered common notation, and an understanding of expressions such as $\sqrt[\pi]{3} = 3^{\frac{1}{\pi}}$ requires more sophisticated techniques. Also, any expression equivalent to 0^0 is not universally defined.

The continuity of the function *f* in focus 2 can be established by its differentiability.