Situation 40: Powers

Prepared at Penn State
Mid-Atlantic Center for Mathematics Teaching and Learning
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Prompt

During an Algebra I lesson on exponents, the teacher asked the students to calculate positive integer powers of 2. A student asks the teacher, “We’ve found 2^2 and 2^3. What about $2^{2.5}$?”

Commentary

This prompt is important because even though this seems like a simple question, it is extending a mathematical operation into other systems beyond the integers. It opens up communication on how to explore such an extension numerically, analytically, and graphically. This situation forces teachers need to go beyond thinking of exponents as repeated multiplication. Investigating the mathematics here will allow one to explain 2^2.5 in a way that makes sense. It is also useful for teachers to consider the fact that f(x) = a^x has several issues for particular values of a and x. In this case, a is a positive number, but if a is negative, the argument in focus 4 does not apply.

Mathematical Foci

Mathematical Focus 1

The value for $2^{2.5}$ can be estimated based on the values for 2^2 and 2^3. It is necessary to understand that the value for $2^{2.5}$ will not be halfway between 2^2 and 2^3.

When investigating the chart of values below, one can talk about the pattern of increasing growth between successive values of 2^x and noting that the growth is different from a linear pattern of growth. Specifically, nonlinearity implies that even though 2.5 is the average of 2 and 3, $2^{2.5}$ will not be the average of 4 and 8.

x	2^x
0	1
1	2
2	4
3	8
4	16
5	32
6	64

Mathematical Focus 2

The value for $2^{2.5}$ can be explored using properties of exponents. In the first case, the expression $2^{2.5}$ can be rewritten as $2^2 2^{0.5} = 2^2 2^{1/2}$ or $2^{5/2}$.

Two raised to the exponent of one-half is equivalent to the square root of 2. Consider the definition for the square root. The square root of a number, r, is a number n such that $n^2 = r$. Thus, if we want to determine the exponent that corresponds to the square root, we can start with the equation $\sqrt{2} = 2^a$. Following the definition for the square root, this would mean that $2^a \cdot 2^a = 2$.

Therefore, $2^{2a} = 2^1$ so $2a = 1$ implying that $a = \frac{1}{2}$.

Using this representation with the properties of exponents, this quantity can be represented by $2^2 \cdot 2^{0.5} = 2^2\sqrt{2} \approx 4(1.414) = 5.656$. So $2^{2.5} \approx 5.656$. Similarly, $\left(2^5\right)^{1/2} = \sqrt{2^5} = \sqrt{32} \approx 5.656$ or $(2^{\frac{1}{2}})^5 = (\sqrt{2})^5 \approx 1.414^5 \approx 5.656$.

Mathematical Focus 3

One possible approach to finding the value of $2^{2.5}$ is to examine the graph of the function $f(x) = 2^x$. It should be recognized that at this point one is using the assumption that the plot can be extended from the discrete case of integer exponents to the continuous case. In particular, there has been a jump from rational to real exponents, and the graph will be represented by a smooth, continuous function.

One can estimate from the graph the value of the function at $x = 2.5$ in at least two different ways. First, one can look at the intersection of the function graph with the vertical line $x = 2.5$ in the following graph to see $f(x) \approx 5.5$. Second, one can trace along the function graph to obtain $f(x) \approx 5.656$ when $x = 2.5$.