Situation 40: Powers

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14 July 2005 – Tracy, Jana, Christa, Jim

Edited at University of Georgia
Center for Proficiency in Teaching Mathematics
19 May 2006 – Eileen, Bob

Prompt

During an Algebra I lesson on exponents, the teacher asked the students to calculate positive integer powers of 2. A student asks the teacher, "We've found 2^2 and 2^3. What about $2^{2.5}$?"

Commentary

This prompt is important because even though this seems like a simple question, it is extending a mathematical operation into other systems beyond the integers. It opens up communication on how to explore such an extension numerically, analytically, and graphically.

Consistency is maintained in all three foci. The choices that are made in the extension of the definition and properties of exponents are made so that they are preserved in a different number system, i.e. from whole numbers to rational numbers. This need for consistency permeates the entire situation and is lying underneath the surface of each focus. For example, in Focus 2, we consider how we can define 2^{rational} in a way that allows the properties of exponents to hold. By deciding that the properties must hold, we can then use them to define the n-th root of 2 as $2^{1/n}$.

This situation forces teachers to go beyond thinking of exponents as repeated multiplication. Investigating the mathematics here will allow one to explain $2^{2.5}$ in a way that makes sense. It is also useful for teachers to consider the fact that f(x) = a^x would behave differently for particular values of "a" and "x". In the discussion below, "a" is assumed to be positive and "x" is a rational number. If "a" had been negative, the arguments would not apply. Furthermore, different mathematical discussions would be necessary if "x" had been irrational, transcendental, or complex.

Mathematical Foci

Mathematical Focus 1

The value for $2^{2.5}$ can be estimated based on the values for 2^2 and 2^3. It is necessary to understand that the value for $2^{2.5}$ will not be halfway between 2^2 and 2^3. When investigating the chart of values below, one can talk about the pattern of increasing growth between successive values of 2^x and noting that exponential growth is different from a linear pattern of growth. Specifically, nonlinearity implies that even though 2.5 is the average of 2 and 3, $2^{2.5}$ will not be the average of 4 and 8.

x	2^x
0	1
1	2
2	4
3	8
4	16
5	32
6	64

In fact, since the first differences are increasing, one can argue that linear approximations will overestimate the value of $2^{2.5}$. Therefore, $2^{2.5}$ should be less than 6.

Visualizing this numerical information is also helpful to see why the value of $2^{2.5}$ will be closer to 4 than to 8.
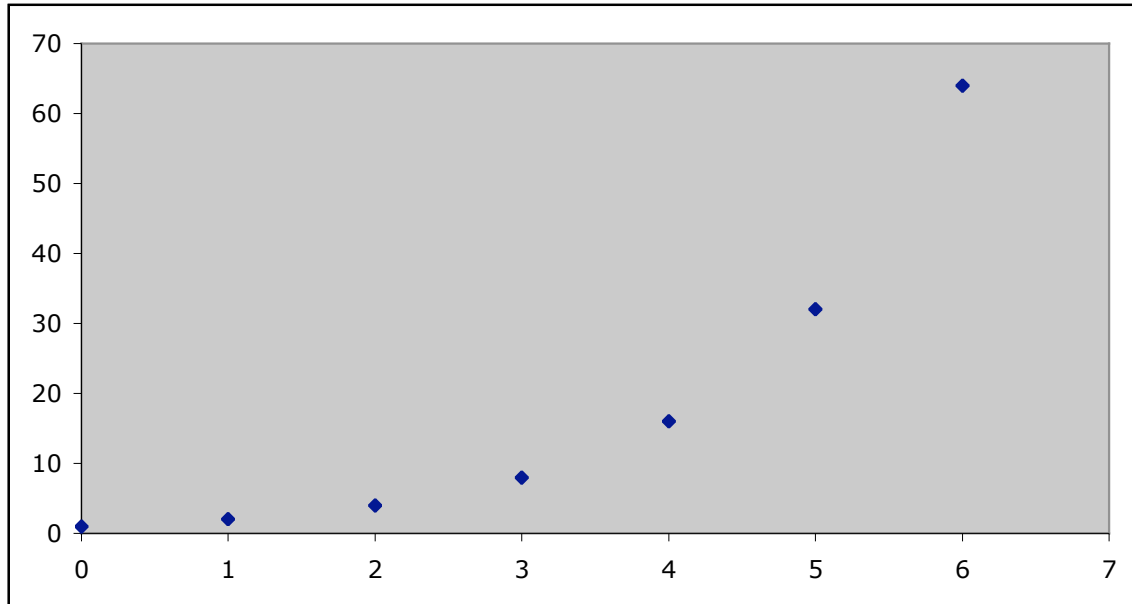
Specifically, if we look at this graph, we can begin to think of a way to connect the points in some consistent way. We can attain this consistency by working with the pattern of increasing growth and realizing that this pattern of points can create a graph that is concave up. We can use this fact to conclude that the value of $2^{2.5}$ will be closer to 4 than to 8.

Mathematical Focus 2

The value for $2^{2.5}$ can be explored using properties of exponents. The expression $2^{2.5}$ can be rewritten as $2^2 2^{0.5} = 2^2 2^{\frac{1}{2}}$ or $2^{\frac{5}{2}}$.

Before we use $2^{\frac{1}{2}}$ in this focus, we feel that it is important to define $2^{\frac{1}{n}}$. Using the properties of exponents, we know that $\underbrace{2^{\frac{1}{n}} \cdot 2^{\frac{1}{n}} \cdot \ldots \cdot 2^{\frac{1}{n}}}_{n \text{ times}} = 2^{\frac{n}{n}} = 2^1 = 2.$

Furthermore, the definition of the root of a number is a number a given integral power of which equals a given number. If the integral power is n, it is the nth root. Thus, $2^{1/n} = \sqrt[n]{2}$. Specifically, $2^{1/2} = \sqrt{2}$.

Using this representation with the properties of exponents, this quantity can be represented by $2^2 \cdot 2^{0.5} = 2^2 \sqrt{2} \approx 4(1.414) = 5.656$. So $2^{2.5} \approx 5.656$. Similarly, $\left(2^5\right)^{1/2} = \sqrt{2^5} = \sqrt{32} \approx 5.656$ or $(2^{\frac{1}{2}})^5 = (\sqrt{2})^5 \approx 1.414^5 \approx 5.656$.

Mathematical Focus 3

One possible approach to finding the value of $2^{2.5}$ is to examine the graph of the function $f(x) = 2^x$. It should be recognized that at this point one is using the assumption that the plot can be extended from the discrete case of integer exponents to the continuous case. In particular, there has been a jump from integer (to rational) to real exponents, and the graph will be represented by a smooth, continuous function.

One can estimate from the graph the value of the function at $x = 2.5$ in at least two different ways. First, one can look at the intersection of the function graph with the vertical line $x = 2.5$ in the following graph to see $f(x) \approx 5.5$. Second, one can trace along the function graph to obtain $f(x) \approx 5.656$ when $x = 2.5$.

no data   Function Plot ▼

— $y = 2^x$
2.5 = 2.5
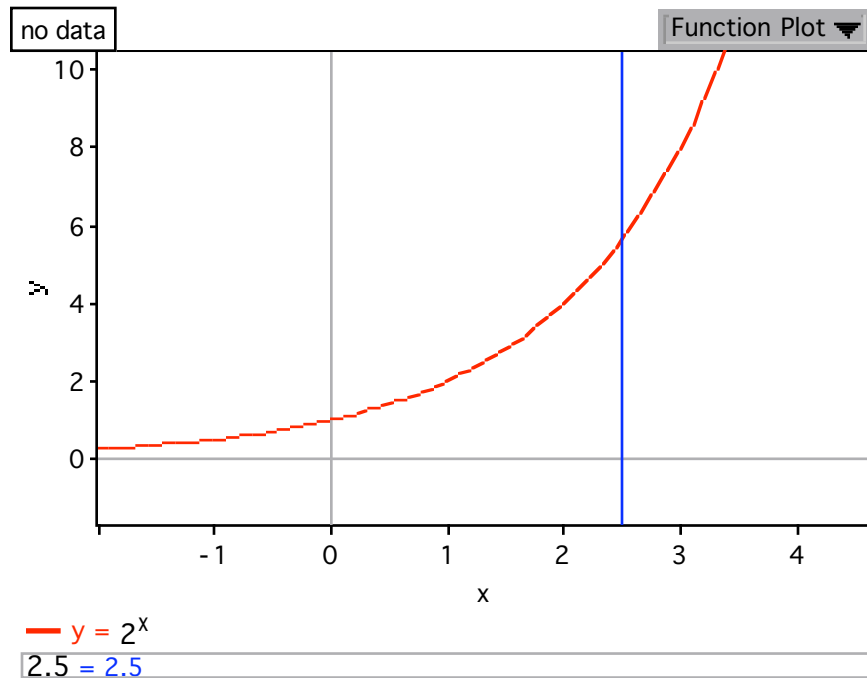
When one uses graphing technology, the technology assumes the solution. However, one could plot the points and use a careful hand or a sophisticated drawing tool to draw a smooth curve to "connect the dots" and the value of $2^{2.5}$ could be approximated from that picture. The concavity of the graph again plays a role in this interpretation of the problem. In particular, when one draws the smooth curve connecting the data points, it is useful to realize that this curve will be concave up.