

## ***Situation 41: Square Roots***

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### **Prompt**

A teacher asks her students to sketch the graph of  $f(x) = \sqrt{-x}$ . A student responds, “That’s impossible! You can’t take the square root of a negative number!”

### **Commentary**

The student’s comment communicates a common misunderstanding that “- $x$ ” always represents a negative number, rather than signifying “opposite of  $x$ .” Teachers must understand this notion of opposites, as well as having a deep understanding of functions: their graphs, reflections, domains, and ranges.

Focus 1 addresses the issue of domain and range of the function, as well as the notion of “- $x$ ” being “opposite of  $x$ ” rather than always a negative number. In Focus 2, the graph of  $f(x) = \sqrt{-x}$  is examined by considering it as a horizontal reflection of  $g(x) = \sqrt{x}$ . Focus 3 presents a numerical approach: create a table of values. Though not a proof, it provides evidence that  $f(x) = \sqrt{-x}$  does, in fact, exist, and that its domain is  $x \leq 0$ . Focus 4 provides another look at the domain and range of a function, and introduces an extension of the domain and range in order to allow for complex numbers, or at least the imaginary numbers.

## Mathematical Foci

### *Mathematical Focus 1*

The domain of the square root function  $f(x) = \sqrt{x}$  is all nonnegative real numbers ( $D: x \geq 0$ ). To find the domain of any square root function, then, one must consider  $x$ -values for which the radicand is greater than or equal to zero. For example, if the function were  $f(x) = \sqrt{x+2}$ , the domain can be found algebraically:

$$x + 2 \geq 0$$

$$x \geq -2$$

In the case of  $f(x) = \sqrt{-x}$ , an algebraic approach for finding the domain is to set  $-x \geq 0$ :

$$-x \geq 0$$

$$x \leq 0$$

In other words, the function  $f(x) = \sqrt{-x}$  does exist and its domain is  $x \leq 0$  (see graph in Focus 2).

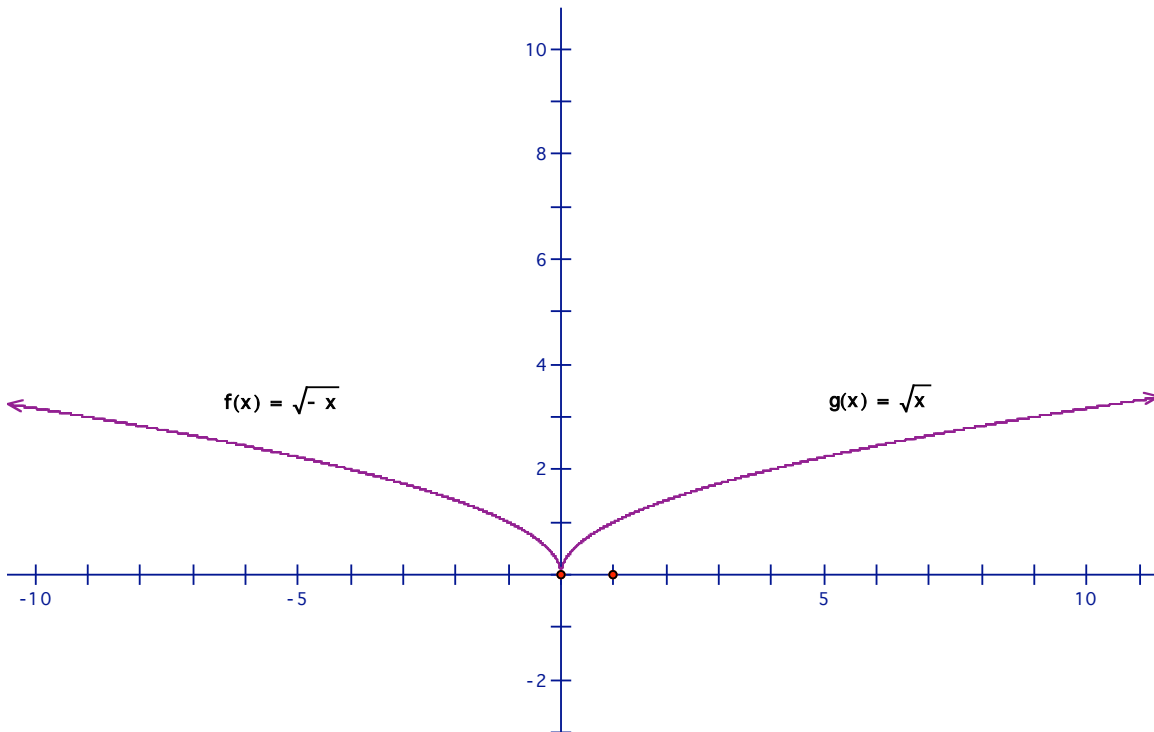
## Mathematical Focus 2

Using a transformation of the graph of the known function  $g(x) = \sqrt{x}$ , the less familiar function,  $f(x) = \sqrt{-x}$ , can be generated. This requires a look into reflections of graphs of functions:

When the graph of  $-h(x)$  is compared to the graph of  $h(x)$ , it can be seen that the two graphs are reflections of each other about the horizontal axis. This is because by graphing the opposite of  $h(x)$ , each point in the positive part of the range of  $h(x)$  becomes negative, and each point in the negative part of the range of  $h(x)$  becomes positive. Since zero has no opposite,  $h(x) = 0$  and  $-h(x) = 0$  are the same on both graphs.

Comparing the graphs of  $h(x)$  and  $h(-x)$  also reveals a reflection: this time the graphs are reflections of each other about the vertical axis. Rather than the range values being negated (as in  $-h(x)$ ), the domain values are negated, resulting in a reflection of  $x$ -values about the vertical axis. Again, since zero has no opposite,  $h(0)$  and  $h(-0)$  are the same point.

Specifically, the graph of the function  $f(x) = \sqrt{-x}$  is a reflection of the graph of  $g(x) = \sqrt{x}$  about the vertical axis, as is shown in the following figure.



The graph illustrates that  $f(x) = \sqrt{-x}$  does exist, and its domain is  $x \leq 0$ .

### Mathematical Focus 3

The results from a numerical approach to  $f(x) = \sqrt{-x}$  echo what has been examined algebraically and graphically. Below is a chart of values of  $f(x) = \sqrt{-x}$  for various  $x$ -values:

$x$	$\sqrt{-x}$
-4	$\sqrt{-(-4)} = 2$
-3	$\sqrt{-(-3)} = \sqrt{3}$
-2	$\sqrt{-(-2)} = \sqrt{2}$
-1	$\sqrt{-(-1)} = 1$
0	$\sqrt{-0} = 0$
1	$\sqrt{-1} = i$
2	$\sqrt{-2} = i\sqrt{2}$
3	$\sqrt{-3} = i\sqrt{3}$
4	$\sqrt{-4} = 2i$

The results show that  $f(x) = \sqrt{-x}$  exists, and suggest that its domain is  $x \leq 0$ . However, when  $x$  is a positive number, the resulting  $f(x)$  is an imaginary number (i.e. does not exist on the real-number coordinate plane). This suggests that teachers need to understand concepts of the system of complex numbers.

## *Mathematical Focus 4*

A concept of domain has been crucial in each of the previous foci. Our discussion for discerning whether the function  $f(x) = \sqrt{-x}$  exists is based on the idea that if we can define a domain (such as  $x \leq 0$ ) for the function, then the function exists. That is, if a function has a domain, then it exists for each value in that domain.

If we consider the possibility that the values of  $x$  and  $f(x)$  could be from the complex numbers, then our domain (and range) could be defined on the set of all complex numbers.

In most cases in school curriculum, it is assumed that when defining the domain of a function, only real number values of  $x$  and  $f(x)$  are considered. There are times, however, when we need to explicitly examine and specify the domain and range. The statement “You can’t take the square root of a negative number” is accurate if the domain is restricted to positive real numbers. It is true that the square root of a negative number is not a real number. But it does exist as a complex (or imaginary) value.

## **Commentary #2**

As mathematics educators, we face the challenge of needing to be precise in our language. The words that we choose are critical, both for mathematical accuracy and to avoid misconceptions among both teachers and students. We cannot be sloppy in our choice of language.

Mathematical terms have precise meanings. The words “negative” and “opposite” have come up in this Situation. “Negative” is a kind of number, while “opposite” describes the relationship of one number to another. It was noted earlier that part of the students’ misconception was that “ $-x$ ” is always a negative number. This no doubt arises from the habit of calling  $-x$  “negative  $x$ .” Perhaps a better (and less confusing) name for  $-x$  is “opposite  $x$ .” This way it is clear that  $x$  could be positive or negative—“ $-x$ ” will simply be its opposite. It is especially important that teachers and students alike understand this difference: “negative” is a kind of number, while “opposite” describes the relationship of one number to another.