# *Situation 41: Square Roots*

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## Prompt

A teacher asks her students to sketch the graph of $f(x) = \sqrt{-x}$. A student responds, "That's impossible! You can't take the square root of a negative number!"

## Commentary

The student's comment communicates a common misunderstanding that "-*x*" always represents a negative number, rather than signifying "opposite of *x*." Teachers must understand this notion of opposites, as well as having a deep understanding of functions: their graphs, reflections, domains, and ranges.

Focus 1 provides an introduction to the domain and range of a function, as well as an extension of the domain and range in order to allow for complex numbers, or at least the imaginary numbers.

Focus 2 addresses the issue of domain and range of the function $f(x) = \sqrt{-x}$, as well as the notion of "-*x*" being "opposite of *x*" rather than always a negative number.

In Focus 3, the graph of $f(x) = \sqrt{-x}$ is examined by considering it as a horizontal reflection of $g(x) = \sqrt{x}$.

Focus 4 presents a numerical approach: create a table of values. Though not a proof, it provides evidence that $f(x)=\sqrt{-x}$ does, in fact, exist, and that its domain is $x \le 0$.

Finally, an Extension is provided to examine more closely the possibility of graphing imaginary numbers in a complex coordinate plane.

As mathematics educators, we face the challenge of needing to be precise in our language. The words that we choose are critical, both for mathematical accuracy and to avoid misconceptions among both teachers and students. We cannot be sloppy in our choice of language.

Mathematical terms have precise meanings. The words “negative” and “opposite” have come up in this Situation. “Negative” is a <u>kind</u> of number, while “opposite” describes the <u>relationship</u> of one number to another. It was noted earlier that part of the students’ misconception was that “$-x$” is always a negative number. This no doubt arises from the habit of calling $-x$ “negative $x$.” Perhaps a better (and less confusing) name for $-x$ is “opposite of $x$.” This way it is clear that $x$ could be positive or negative—“$-x$” will simply be its opposite. It is especially important that teachers and students alike understand this difference: “negative” is a <u>kind</u> of number, while “opposite” describes the <u>relationship</u> of one number to another.

# <u>Mathematical Foci</u>

## *Mathematical Focus 1*

A concept of domain is crucial in each of the following foci. Our discussion for discerning whether the function $f(x) = \sqrt{-x}$ exists is based on the idea that if we can define a domain (such as $x \le 0$) for the function, then the function exists. That is, if a function has a domain, then it exists for each value in that domain.

If we consider the possibility that the values of $x$ and $f(x)$ could be from the complex numbers, then our domain (and range) could be defined on the set of all complex numbers.

In most cases in school curriculum, it is assumed that when defining the domain of a function, only real number values of $x$ and $f(x)$ are considered. There are times, however, when we need to explicitly examine and specify the domain and range. The statement "You can't take the square root of a negative number" is accurate if the domain is restricted to positive <u>real</u> numbers. It is true that the square root of a negative number is not a <u>real</u> number. But it does exist as a complex (or imaginary) value.

Since complex numbers typically appear later in school curriculum than do square root functions, we will limit our discussion in the remaining Foci to real-number values for domain and range. Complex numbers are considered in the Extension.

## *Mathematical Focus 2*

If we do only consider real values of $x$ and $f(x)$, then the domain of the square root function $f(x) = \sqrt{x}$ is all nonnegative real numbers ($D$: $x \geq 0$). To find the domain of any square root function, then, one must consider $x$-values for which the radicand is greater than or equal to zero. For example, if the function were $f(x) = \sqrt{x+2}$, the domain can be found algebraically:

$$x + 2 \geq 0$$
$$x \geq -2$$

In the case of $f(x) = \sqrt{-x}$, an algebraic approach for finding the domain is to set $-x \geq 0$:

$$-x \geq 0$$
$$x \leq 0$$

In other words, the function $f(x) = \sqrt{-x}$ does exist and its domain is $x \leq 0$ (see graph in Focus 3).

Note: The last step in the algebraic manipulation above demonstrates an important rule when working with inequalities: when multiplying (or dividing) by a negative number (such as -1 in this case), the inequality symbol must be flipped. For example, the statement $2 < 5$ is true, but if the terms are multiplied by -1, then the resulting expression, $-2 < -5$, is no longer true. The inequality must be flipped: $-2 > -5$.

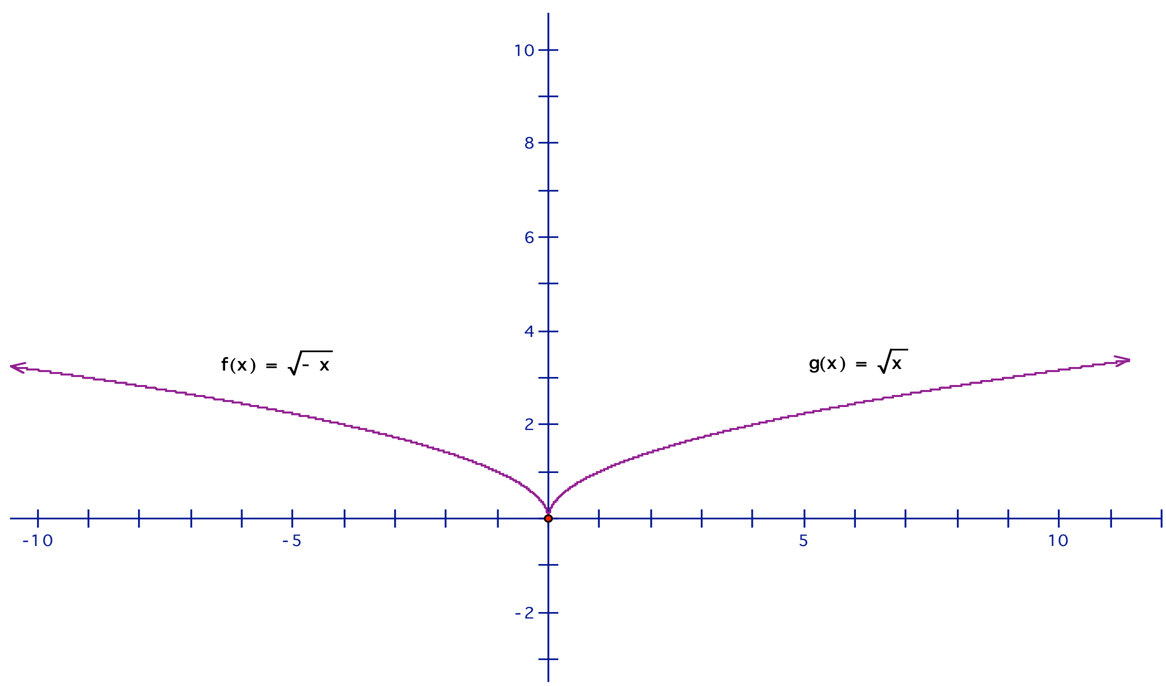
*Mathematical Focus 3*

In this Focus, we shall again consider only real input and output values. That is, we shall graph the functions on a coordinate plane of real values. Using a transformation of the graph of the known function $g(x) = \sqrt{x}$, the less familiar function, $f(x) = \sqrt{-x}$, can be generated. This requires a look into reflections of graphs of functions:

When the graph of -*h*(*x*) is compared to the graph of *h*(*x*), it can be seen that the two graphs are reflections of each other about the horizontal axis. This is because by graphing the opposite of *h*(*x*), each point in the positive part of the range of *h*(*x*) (i.e. above the *x*-axis) becomes negative, and each point in the negative part of the range of *h*(*x*) becomes positive. Since zero has no opposite, *h*(*x*) = 0 and -*h*(*x*) = 0 are the same on both graphs.

Comparing the graphs of *h*(*x*) and *h*(-*x*) also reveals a reflection: this time the graphs are reflections of each other about the vertical axis. Rather than the range values being negated (as in -*h*(*x*)), the domain values are negated, resulting in a reflection of *x*-values about the vertical axis. Again, since zero has no opposite, *h*(0) and *h*(-0) are the same point.

Specifically, the graph of the function $f(x) = \sqrt{-x}$ is a reflection of the graph of $g(x) = \sqrt{x}$ about the vertical axis, as is shown in the following figure.



The graph illustrates that $f(x) = \sqrt{-x}$ does exist, and that its domain is $x \le 0$.

*Mathematical Focus 4*

The results from a numerical approach to $f(x) = \sqrt{-x}$ echo what has been examined algebraically and graphically. Below is a chart of values of $f(x) = \sqrt{-x}$ for various *x*-values:

| x | $\sqrt{-x}$ |
|---|---|
| -4 | $\sqrt{-(-4)} = 2$ |
| -3 | $\sqrt{-(-3)} = \sqrt{3}$ |
| -2 | $\sqrt{-(-2)} = \sqrt{2}$ |
| -1 | $\sqrt{-(-1)} = 1$ |
| 0 | $\sqrt{-0} = 0$ |
| 1 | $\sqrt{-1} = i$ |
| 2 | $\sqrt{-2} = i\sqrt{2}$ |
| 3 | $\sqrt{-3} = i\sqrt{3}$ |
| 4 | $\sqrt{-4} = 2i$ |

The results show that $f(x) = \sqrt{-x}$ exists, and suggest that its domain is $x \le 0$. However, when $x$ is a <u>positive</u> number, the resulting $f(x)$ is an imaginary number (i.e. does not exist on the real-number coordinate plane).

## *Extension: Complex-Numbers Coordinate Plane*

If we are not limited to real numbers, the possibilities for $f(x) = \sqrt{-x}$ are much greater than when we have the real-number restriction. That is, if a value such as $\sqrt{-4}$, or 2*i*, were "allowed" as a member of the range of *f*(*x*), then the domain would no longer be restricted to non-positive values.

One way to graph complex numbers on a coordinate plane is to have one axis correspond to the real number parts, and the other axis correspond to the imaginary parts, of complex numbers. So, for example, 2 + 3*i* would be graphed as a point in the first quadrant: