MAC-CPTM Situations Project

Situation 41: Square Roots

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Prompt

A teacher asked her students to sketch the graph of $f(x) = \sqrt{-x}$. A student responded, “That’s impossible! You can’t take the square root of a negative number!”

Commentary

This situation addresses several key concepts that occur frequently in school mathematics: opposites of numbers, negative numbers, function, domain, and ranges. Since the symbol “-“ has multiple interpretations, it is important to distinguish between a negative number and the opposite of a number. Moreover, the domain over which a function is defined determines the range over which the function is defined, and a table of values provides an example to illustrate the relationship between domain and range. For a set of points with coordinates $\left(x, f(x)\right)$ to define the graph of a function, each first coordinate, x, must correspond to a unique second coordinate, *f*(x). A graphical representation highlights the univalent relationship between x and *f*(x).

Mathematical Foci

Mathematical Focus 1

In Algebra, - x is a notation that represents the opposite of x

Mathematical terms have precise meanings. The symbol "-" is commonly read as both negative and opposite. However, a negative number is a *kind* of number, while the opposite of a number describes the *relationship* of one number to another. For example, negative 6 (-6) indicates a number <0, and the number opposite of positive 6 (+6) indicates the additive inverse of +6, which is negative 6 (-6). Using the variable x to represent a number does not indicate the kind of number (e.g. positive, negative, zero). In this way, –x represents the opposite or additive inverse of x, which could be positive, negative, or zero.

Mathematical Focus 2

The domain of a function is critical in determining the values over which the range of a function is defined.

The implicit assumption that the domain and range of a function are restricted to real numbers could contribute to the statement "You can't take the square root of a negative number." If the domain and range of the function *f* with rule $f(x)=\sqrt{-x}$ are restricted to real numbers, then *f* is defined only for x≤0. Similarly, if the domain and range of the function *h* with rule $h(x)=\sqrt{x+2}$ are restricted to real numbers, then *h* is defined only for x ≥ -2. If the domain of the function *f* with rule $f(x)=\sqrt{-x}$ includes all real numbers, then the range of *f* is a set of complex numbers.

The table below provides an example that illustrates how the domain of the function *f* with rule $f(x)=\sqrt{-x}$ determines values over which the range of *f* is defined:

x	$\sqrt{-x}$
-4	$\sqrt{-(-4)}=2$
-3	$\sqrt{-(-3)}=\sqrt{3}$
-2	$\sqrt{-(-2)}=\sqrt{2}$
-1	$\sqrt{-(-1)}=1$
0	$\sqrt{-0}=0$
1	$\sqrt{-1}=i$
2	$\sqrt{-2}=i\sqrt{2}$
3	$\sqrt{-3}=i\sqrt{3}$
4	$\sqrt{-4}=2i$

The values in the table above are consistent with the ordered pairs for the function *f* with rule $f(x)=\sqrt{-x}$. If x≥0, the radicand ≥0, and the range of *f* is the positive real numbers, including zero. If x>0, the radicand <0, and the range of *f* is a set of complex numbers.

Mathematical Focus 3

A reflection in the vertical axis maps a point with coordinates $\left(x,\sqrt{x}\right)$ *to a corresponding point with coordinates* $\left(-x,\sqrt{x}\right)$. *To prove that a set of points is the graph of a function, prove that each first coordinate within the set of points has a unique second coordinate.*

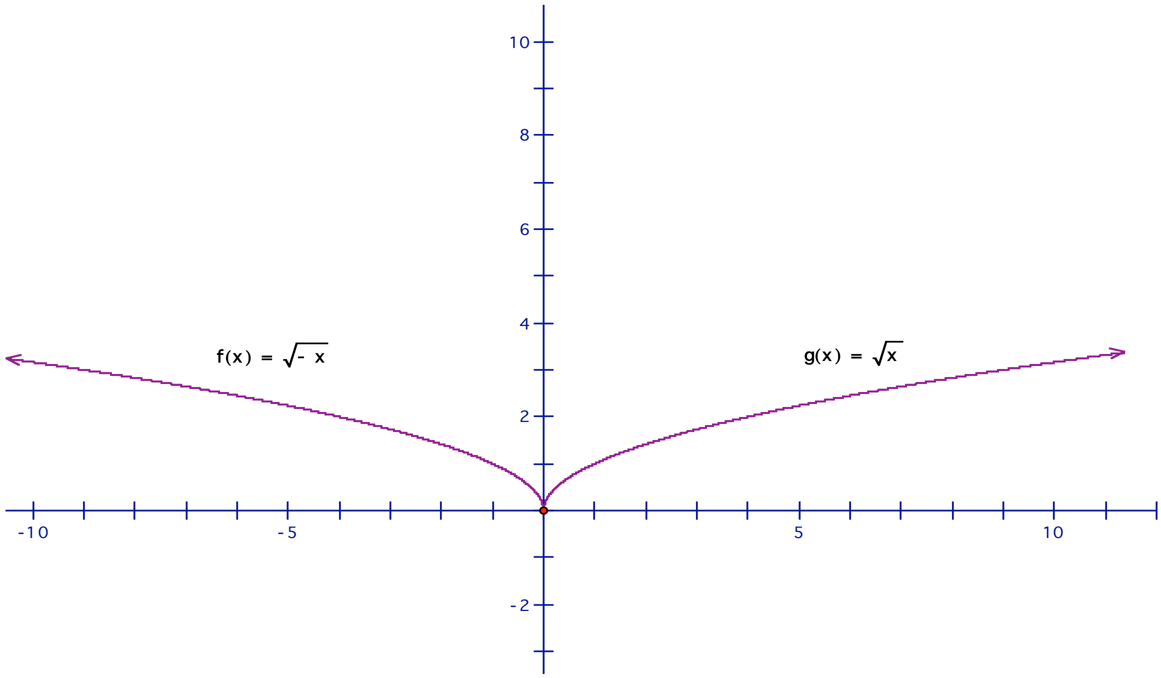
In general, a reflection in the vertical axis maps the point with coordinates (x,y) to a corresponding point with coordinates (-x,y).

The graph of the function g with domain x≥0 and rule $g(x)=\sqrt{x}$ is given by the set of points with coordinates $\left(x,g(x)\right)$, or equivalently, $\left(x,\sqrt{x}\right)$. Reflecting in the vertical axis maps the set of points with coordinates $\left(x,g(x)\right)$ with x≥0 to the set of points with coordinates $\left(-x,g(x)\right)$ with x≥0, or equivalently, $\left(-x,\sqrt{x}\right)$ with x≥0.

Since the set of points with coordinates $\left(x,g(x)\right)$ defines the graph of the function g with domain x≥0 and rule $g(x)=\sqrt{x}$, each first coordinate, x, corresponds to a unique second coordinate, $g(x)=\sqrt{x}$. Since reflecting in the vertical axis maps each first coordinate, x, to a corresponding first coordinate, –x, each first coordinate, –x, will also correspond to a unique second coordinate, $g(x)=\sqrt{x}$. In this way, the set of points with coordinates $\left(-x,\sqrt{x}\right)$ with x≥0 also defines the graph of a function.

Since –x represents the opposite of x, when x>0, -x is negative and when x<0, -x is positive. In this way, the set of points with coordinates $\left(x,\sqrt{-x}\right)$ with x≤0 is the same as the set of points with coordinates $\left(-x,\sqrt{x}\right)$ with x≥0. Moreover, the set of points with coordinates $\left(x,\sqrt{-x}\right)$ with x≤0 defines the graph of the function *f* with domain x≤0 and rule $f(x)=\sqrt{-x}$.

Graphs of the functions *f* and *g* are shown below.


f(x) = √- x
g(x) = √x
10
8
6
4
2
-2
-10
-5
5
10