

Situation: Can You Circumscribe a Circle about This Polygon?

Prepared at Penn State

Mid-Atlantic Center for Mathematics Teaching and Learning

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Prompt

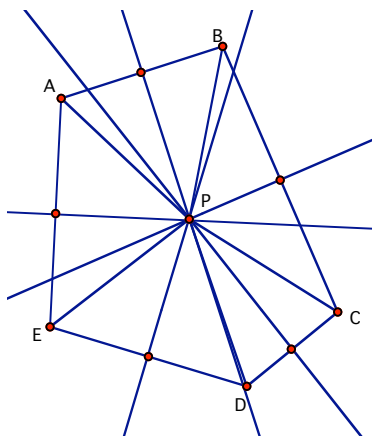
In a geometry class, after a discussion about circumscribing circles about triangles, a student asks, “Can you circumscribe a circle about any polygon?”

Commentary

Mathematical Foci

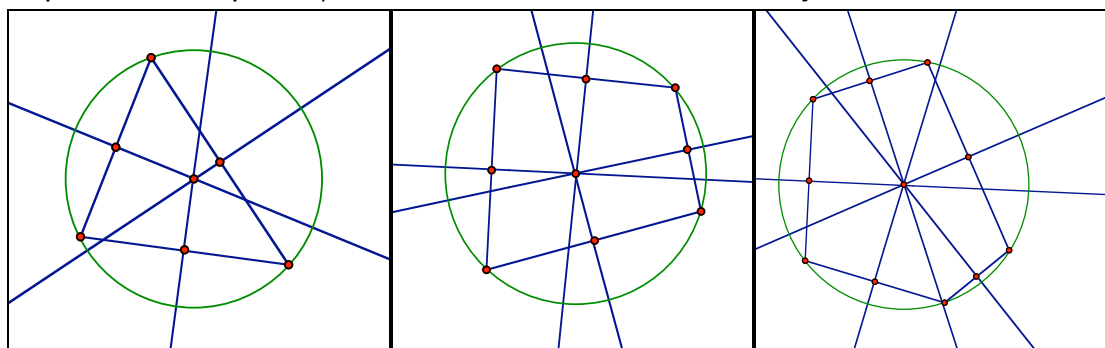
Mathematical Focus 1

Circumscribing a circle about a triangle is accomplished by finding the circumcenter of the triangle. The circumcenter of a triangle is the point that is the center of the circle that passes through the triangle’s three vertices. It is the point at which the three perpendicular bisectors of the sides of the triangle are concurrent. Since every triangle has a circumcenter, it is possible to circumscribe a circle about any triangle (see proof in Appendix A) by using the circumcenter as the center and the distance from the circumcenter to a vertex as the radius. Thus it is easy to see what might have prompted the student’s question. A criterion by which one can judge whether a polygon can be circumscribed by a circle is based on the perpendicular bisectors of the sides of the polygon. If the perpendicular bisectors of a polygon are concurrent, then one can circumscribe a circle about that polygon, and the polygon is called a cyclic polygon. If the perpendicular bisectors of a polygon are concurrent, then we know that their point of concurrence is equidistant from the vertices of the polygon. This is because every point on the perpendicular bisector of a segment is equidistant from the endpoints of that segment. For example, consider pentagon ABCDE with perpendicular bisectors concurrent at P.



$AP = BP$ since P is on the perpendicular bisector of \overline{AB} ; $BP = CP$ since P is on the perpendicular bisector of \overline{BC} ; $CP = DP$ since P is on the perpendicular bisector of \overline{CD} ; $DP = EP$ since P is on the perpendicular bisector of \overline{DE} ; and $EP = AP$ since P is on the perpendicular bisector of \overline{EA} . Then, by the transitive property of equality, $AP = BP = CP = DP = EP$, and the point of concurrence of the perpendicular bisectors is equidistant from the vertices of the pentagon. A similar argument can be made for any polygon for which the perpendicular bisectors of the sides are concurrent.

The figures below, also available dynamically in SIT P 050802 PolyCircT.gsp, show various polygons with their circumcircles. Every regular polygon is cyclic (Clapham, 1996, p. 214) and thus can be circumscribed by a circle.



Triangle

Quadrilateral

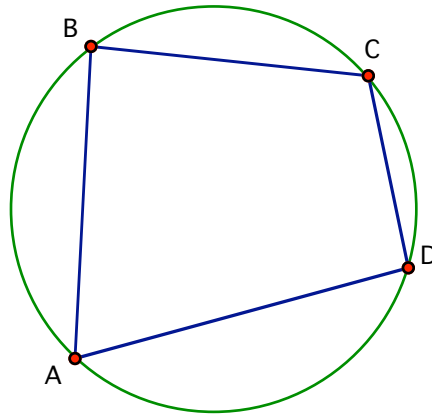
Pentagon

Mathematical Focus 2

A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.

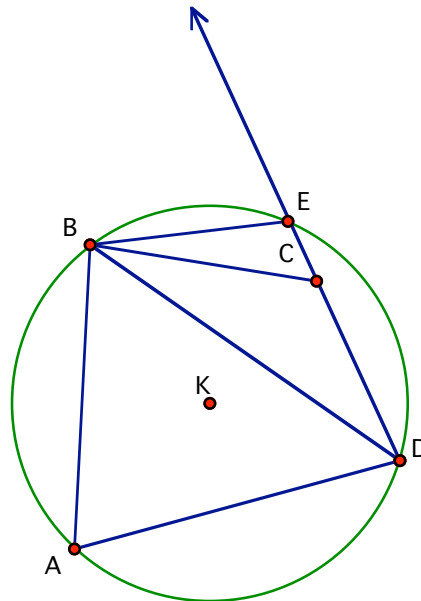
We can prove that a quadrilateral inscribed in a circle has opposite angles that are supplementary by observing that the two arcs in which the opposite angles are inscribed together form the entire circle, so the sum of the degree measures in the two arcs is 360. For example, in quadrilateral $ABCD$ below, $\angle ABC$ is opposite $\angle CDA$ and arcs CDA and ABC form the entire circle. Since the

measure of an inscribed angle is half of the measure of the arc in which it is inscribed, the sum of the measures of the two angles must be 180° . Thus the angles are supplementary.



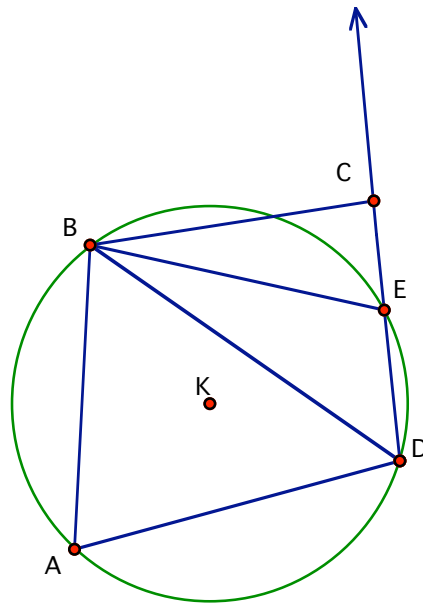
Quadrilateral ABCD

A proof that if the opposite angles of a quadrilateral are supplementary, then it can be inscribed in a circle can be accomplished indirectly. Given the quadrilateral ABCD, with $m\angle A + m\angle C = 180^\circ$, construct circumcircle K for triangle ABD. First, suppose point C is in the interior of the circle and let E be the intersection of \overrightarrow{DC} with circle K. Then ABED is inscribed in circle K and $\angle E$ is supplementary to $\angle A$ as proved above. But then both $\angle E$ and $\angle C$ are supplementary to $\angle A$, so $\angle E$ is congruent to $\angle C$. This implies that \overline{BC} is parallel to \overline{BE} , which is a contradiction.



Quadrilaterals ABCD and ABED with C in the interior of the circle

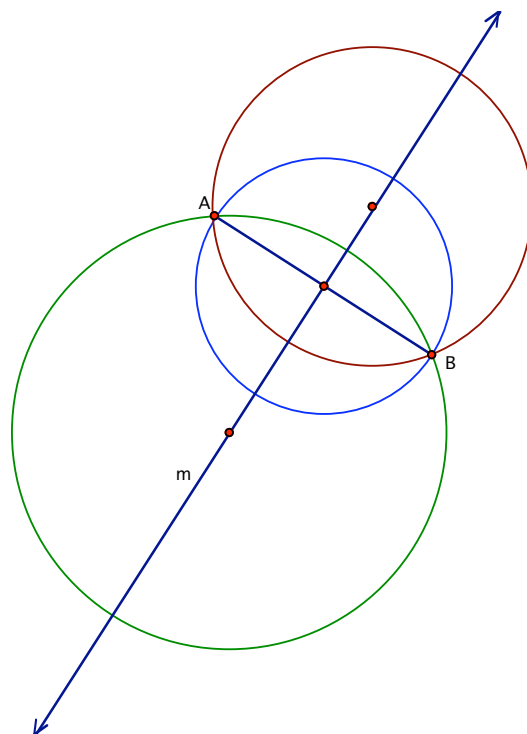
On the other hand, suppose point C is in the exterior of the circle. By the same argument, we have \overline{BC} is parallel to \overline{BE} , which is a contradiction. Thus point C must be on circle K, and circle K is circumscribed about quadrilateral ABCD.



Quadrilaterals ABCD and ABED with C in the exterior of the circle

Mathematical Focus 3

Another way to think about this problem is to ask, “Can you draw a circle through any given number of points?” Many circles can be drawn through any two points; the center of a circle through any two points should be on the perpendicular bisector of the segment joining the points. (In the figure below, m is the perpendicular bisector of segment AB. Three circles are drawn for illustration. A dynamic illustration of this is available in the page called “Two points” in SIT P 050817 PolyCircC.gsp.)



Three of the circles through points A and B

This constrains the center to lie anywhere on a line, and then the radius is constrained by the choice of the center. A unique circle can be drawn through any three noncollinear points; its center is constrained to be the intersection of the perpendicular bisectors of segments joining the points. No circle can be drawn through three distinct collinear points. If you have more than three points, in order to draw a circle through all of them, the additional points must fall on the circle already determined by three of them. If not, there is no circle through all the points. Thus you can draw a circle through more than three points only in very special circumstances, namely, when the perpendicular bisectors of the sides of the polygon formed by the points are concurrent.

The need for the remaining points to fall on the circle determined by three of the points is similar to the requirements for the collinearity of a set of points. Since two points determine a line, a line can be drawn through three points only if the third falls on the line already determined by the first two.

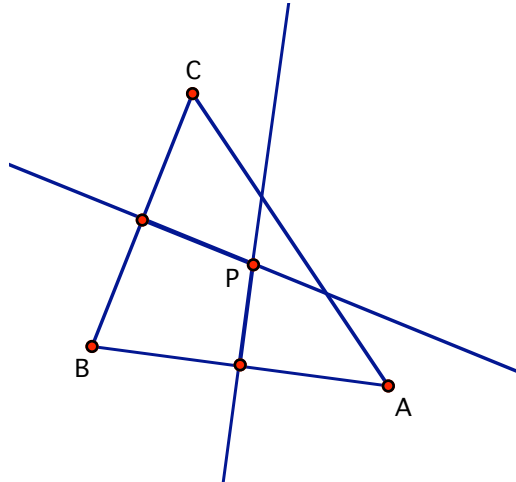
References

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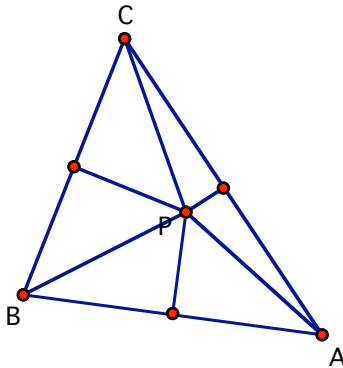
Appendix A

The following is a proof that the perpendicular bisectors of a triangle intersect in a point and that point is equidistant from A, B, and C.

Given $\triangle ABC$ and the perpendicular bisectors of \overline{AB} , \overline{BC} , and \overline{AC} , we know that sides \overline{BC} and \overline{AC} cannot be parallel, therefore segments perpendicular to those sides cannot be parallel. So the perpendicular bisectors must intersect at some point, P.



Through any two points there is exactly one line, therefore we can draw segments \overline{PA} , \overline{PB} , and \overline{PC} .



A point is on the perpendicular bisector of a segment if and only if it is equidistant from the endpoints of the segment. Therefore, $PA = PC$ and $PC = PB$. By the Transitive Property of Equality, $PA = PB$. P is on the perpendicular bisector of AB, because it is equidistant from the endpoints of the segment. Thus, we have established that P is equidistant from the vertices of the triangle and the perpendicular bisectors of a triangle are concurrent.

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