MAC-CPTM Situations Project

Situation 43: Can You Circumscribe a Circle about This Polygon?

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Prompt

In a geometry class, after a discussion about circumscribing circles about triangles, a student asks, "Can you circumscribe a circle about any polygon?"

Commentary

The conditions under which a circle can be found to circumscribe a give polygon are related to the relationships among angles, sides, and the perpendicular bisectors of the sides of polygons.

Mathematical Foci

Mathematical Focus 1

A unique circle can be drawn through any three noncollinear points in a plane; its center is constrained to be the intersection of the perpendicular bisectors of segments joining the points.

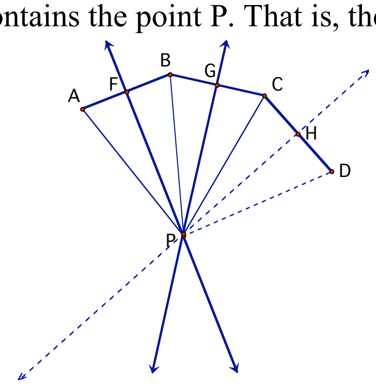
Circumscribing a circle about a triangle requires finding a point that is equidistant from the three vertices of the triangle, or, equivalently, finding the circumcenter of the triangle. The circumcenter of a triangle is the center of the circle that passes through the triangle's three vertices. Because perpendicular bisectors are lines each of whose points are equidistant from the endpoints of a segment, it makes sense to consider perpendicular bisectors in a search for the circumcenter. Given ΔABC and the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$, D and E, respectively. $\overline{AB}$ and $\overline{BC}$ are not parallel–so lines that are perpendicular to them are not parallel. So the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ must intersect at some point, call it P. P is equidistant from A and B because it lies on the perpendicular bisector of $\overline{AB}$, and P is equidistant from B and C because it lies of the perpendicular bisector of $\overline{BC}$. So P is equidistant from A, B, and C–that is, P is the circumcenter of ΔABC.

Mathematical Focus 2

Every regular polygon is cyclic and thus can be circumscribed by a circle.
By examining the perpendicular bisectors of the sides of a polygon, one can determine conditions that are sufficient to conclude that a circle can circumscribe the polygon. If one can circumscribe a circle about a polygon, the polygon is called a cyclic polygon. Thus, since one can circumscribe a circle about any triangle, every triangle is a cyclic polygon.

Locating a circumcenter for other polygons can be facilitated by consideration of perpendicular bisectors. Every point on the perpendicular bisector of a segment is equidistant from the endpoints of that segment. This fact supports the conclusison that if the perpendicular bisectors of a polygon are concurrent, the polygon is cyclic. The intersection point, *P*, of the perpendicular bisectors of two adjacent sides of a polygon, say $\overline{AB}$ and $\overline{BC}$, is equidistant from the vertices defining the sides (in this case, *A*, *B*, and *C*). Because $\overline{PA} = \overline{PB}$ and $\overline{PB} = \overline{PC}$ the intersection point, P, is the same distance from *A*, *B,* and *C*. This argument can be extended to any n-sided polygon for which the perpendicular bisectors of the sides are concurrent to show that if the perpendicular bisectors of a polygon are concurrent, then their point of concurrence, or circumcenter, is equidistant from the vertices of the polygon.

The question remains as to which polygons have concurrent perpendicular bisectors. It can be shown that the perpendicular bisectors of a regular polygon are concurrent. By definition, a regular polygon is an equilateral and equiangular n-sided polygon. Consider a regular polygon with adjacent vertices, A, B, C, and D. Let P be the point of intersection of the perpendicular bisectors ($\overline{FP}$ and $\overline{GP}$, respectively) of $\overline{AB}$ and $\overline{BC}$. It can be shown that $\triangle AFP \cong \triangle BFP \cong \triangle BGP \cong \triangle CGP$ using the fact that P is equidistant from A, B, and C, and using the HL (hypotenuse-leg) theorem. Construct $\overline{PH}$ perpendicular to $\overline{DC}$ and consider $\triangle HCP$. It can be shown that $\triangle FBP \cong \triangle HCP$. Because HC=FB, FB=AF, and the polygon is regular, it follows that HC=HD. And so $\overline{PH}$ is the perpendicular bisector of $\overline{DC}$. The argument can be extended to successive vertices of the polygon, resulting in establishing that each of the perpendicular bisectors of the sides contains the point P. That is, the perpendicular bisectors are concurrent.



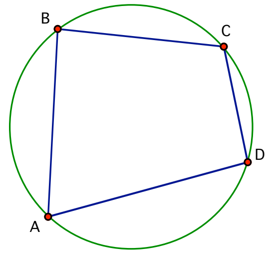
Therefore, every regular polygon is cyclic and thus can be circumscribed by a circle.

Mathematical Focus 3

A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.

Given a quadrilateral inscribed in a circle, it can be proved that the opposite angles are supplementary. Observe that the two arcs in which the opposite angles are inscribed together form the entire circle, so the sum of the degree measures in the two arcs is 360°.

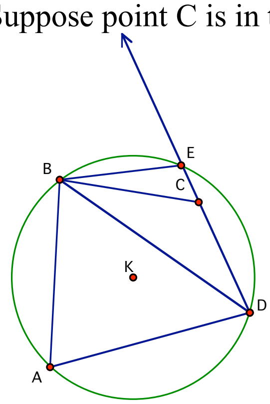
For example, in quadrilateral ABCD, shown in the following diagram, ∠ABC is opposite ∠CDA and arcs CDA and ABC form the entire circle. Since the measure of an inscribed angle is half of the measure of the arc in which it is inscribed, the sum of the measures of the two angles must be 180°. Thus the angles are supplementary.



Quadrilateral ABCD

Given that the opposite angles of a quadrilateral are supplementary, it can be indirectly proved that the quadrilateral can be inscribed in a circle. First, construct circumcircle K for triangle ABD, and then construct quadrilateral ABCD, such that m∠BAD + m∠BCD = 180. By showing that it is impossible for point C to be either in the interior or the exterior of circumcircle K, it can be concluded that point C must be on circumcircle K and hence, quadrilateral ABCD can be inscribed in a circle.

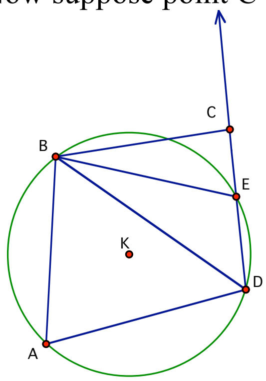
Suppose point C is in the interior of the circle, as shown in the diagram below.



Quadrilaterals ABCD and ABED with C in the interior of the circle

If point C is in the interior of the circle, then point E would be the intersection of $\overrightarrow{DC}$ with circle K. Then quadrilateral ABED would be inscribed in circle K and ∠BED would also be supplementary to ∠BAD. But then both ∠BED and ∠BCD would be supplementary to ∠BAD, so ∠BED would be congruent to ∠BCD, implying that $\overline{BC}$ would be parallel to $\overline{BE}$, which is impossible. Hence, point C cannot be in the interior of circle K.

Now suppose point C is in the exterior of the circle, as shown in the diagram below.



Quadrilaterals ABCD and ABED with C in the exterior of the circle

By the same argument, $\overline{BC}$ would have to be parallel to $\overline{BE}$, which is impossible.

Thus point C must be on circle K, and quadrilateral ABCD is inscribed in circle K. In this way, the center of circumcircle K will be the circumcenter of triangle ABD, triangle BCD, and quadrilateral ABCD. Therefore, the point of concurrence of the perpendicular bisectors of $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{AD}$ will be the center of circumcircle K.