

MAC-CPTM Situations Project

Situation 43: Can You Circumscribe a Circle about This Polygon?

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Prompt

In a geometry class, after a discussion about circumscribing circles about triangles, a student asks, “Can you circumscribe a circle about any polygon?”

Commentary

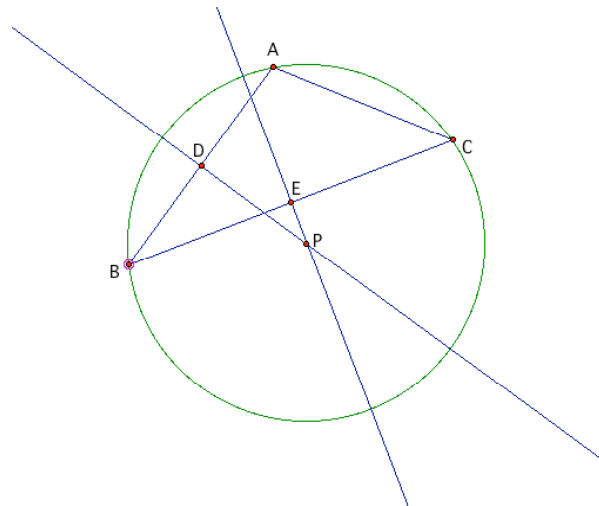
A polygon that can be circumscribed by a circle is called a cyclic polygon. Not every polygon is circumscribable. The conditions under which a circle circumscribes a given polygon are dependent upon the relationships among the angles, the sides, and the perpendicular bisectors of the sides of the polygon. While not every polygon is cyclic, certain classes of polygons are cyclic if they satisfy specific conditions. Each focus describes such classes and Focus 1 provides one way to check whether a given polygon is circumscribable.

Mathematical Foci

Mathematical Focus 1

Every triangle is cyclic. This fact is core to establishing a condition for other polygons to be cyclic.

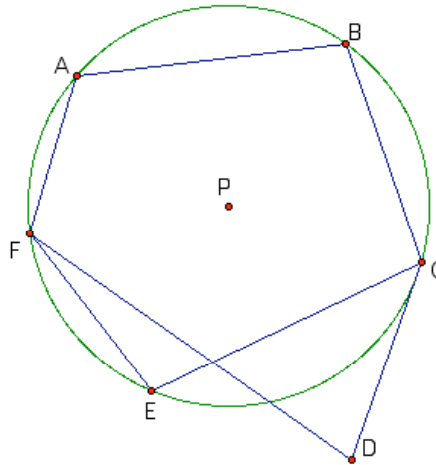
Circumscribing a circle about a triangle requires finding a point that is equidistant from the three vertices of the triangle, or, equivalently, finding the circumcenter of the triangle. The circumcenter of a triangle is the center of the circle that passes through the triangle's three vertices. Because perpendicular bisectors are lines each of whose points is equidistant from the endpoints of a segment, it makes sense to consider perpendicular bisectors of the sides of a triangle to find the circumcenter. Given $\triangle ABC$ and the perpendicular bisectors (in the plane of $\triangle ABC$) of \overline{AB} and \overline{BC} , D and E , respectively. \overline{AB} and \overline{BC} are not parallel—so lines that are perpendicular to them are not parallel. So the perpendicular bisectors of \overline{AB} and \overline{BC} must intersect at some point, call it P . P is equidistant from A and B because it lies on the perpendicular bisector of \overline{AB} , and P is equidistant from B and C because it lies on the perpendicular bisector of \overline{BC} . So P is equidistant from A , B , and C —that is, P is the circumcenter of $\triangle ABC$.



So, given any set of three points, we can find a circumcenter. It remains to be shown that a unique circle that circumscribes these points. Suppose there is a circumcenter, Q , different from P —and hence a different circle that circumscribes A , B , and C . If Q is a circumcenter for A , B , and C , then Q is equidistant from A , B , and C . Using triangle congruence relationships, we can establish that Q is on the perpendicular bisector of the segment AB and that Q is on the perpendicular bisector of the segment BC . Since two distinct lines intersect in exactly one point, and since P is also on the perpendicular bisectors of the segment AB and the segment BC , then P and Q are not distinct. Thus the circumcenter is unique and the circumcircle is unique.

For a polygon with more than three vertices to be circumscribable, its vertices must all lie on that unique circle. For example, the polygon $ABCEF$ is cyclic, since vertices E and F

lie on the circle defined by A , B , and C . However, polygons with vertices that do not lie on the circle defined by three of its vertices are not circumscribable. For example, the polygon $ABCDF$ has one vertex (D), which does not lie on the circle defined by A , B , and C , and thus the polygon is not cyclic.



Mathematical Focus 2

Every regular polygon is cyclic. However, not every cyclic polygon is regular.

Every triangle is a cyclic polygon. The question remains as to which other polygons are cyclic. By examining the perpendicular bisectors of the sides of a polygon, one can determine a set of conditions on a polygon that is sufficient to conclude whether a circle can circumscribe that polygon. In particular, we can show that every regular polygon can be circumscribed by a circle—that is, every regular polygon is cyclic.

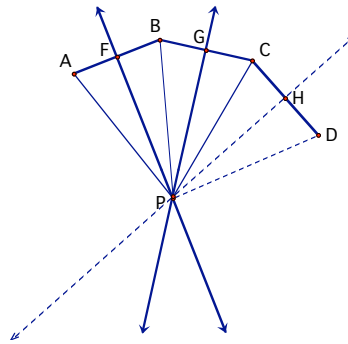
To show that every regular polygon is cyclic, we first will establish that if the perpendicular bisectors of the sides of a polygon are concurrent, the polygon is cyclic. We will then show that the perpendicular bisectors of the sides of a regular polygon are concurrent and conclude that every regular polygon is cyclic.

Consider a polygon of an arbitrary number of sides. Our argument depends on choosing and analyzing distance relationships for adjacent (arbitrarily chosen) vertices. Locating a circumcenter for polygons can be facilitated by consideration of perpendicular bisectors. Every point on the perpendicular bisector of a segment is equidistant from the endpoints of that segment. This fact supports the conclusion that if the perpendicular bisectors of the sides of a polygon are concurrent, the polygon is cyclic as follows. The intersection point, P , of the perpendicular bisectors of two adjacent sides of a polygon, say \overline{AB} and \overline{BC} , is equidistant from the vertices defining the sides (in this case, A , B , and C).

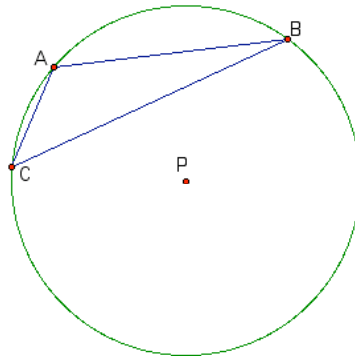
Because $\overline{PA} = \overline{PB}$ and $\overline{PB} = \overline{PC}$ the intersection point, P , is the same distance from A , B , and C . This argument can be extended to any n -sided polygon for which the perpendicular bisectors of the sides are concurrent to show that if the perpendicular

bisectors of a polygon are concurrent, then their point of concurrence is equidistant from the vertices of the polygon. We call this point of concurrence the circumcenter.

The remaining goal is to show that the perpendicular bisectors of the sides of a regular polygon are concurrent. By definition, a regular polygon is an equilateral and equiangular n -sided polygon. Consider a regular polygon with adjacent vertices, A , B , C , and D . Let P be the point of intersection of the perpendicular bisectors (\overline{FP} and \overline{GP} , respectively) of \overline{AB} and \overline{BC} . It can be shown that $\triangle VAFP \cong \triangle VBFP \cong \triangle VBGP \cong \triangle VCGP$ using the fact that P is equidistant from A , B , and C , and using the HL (hypotenuse-leg) theorem. Construct \overline{PH} perpendicular to \overline{DC} and consider $\triangle VHCP$. It can be shown that $\triangle VFBP \cong \triangle VHCP$. Because $HC=FB$, $FB=AF$, and the polygon is regular, it follows that $HC=HD$. And so \overline{PH} is the perpendicular bisector of \overline{DC} . The argument can be extended to successive vertices of the polygon, resulting in establishing that each of the perpendicular bisectors of the sides contains the point P . That is, the perpendicular bisectors are concurrent. Note that this argument does break down if the polygon in question is not regular. If the polygon is not regular (specifically if the sides are not equal to each other) we cannot prove that any of the triangles $\triangle VAFP$, $\triangle VBFP$, $\triangle VBGP$, $\triangle VCGP$, $\triangle VHCP$ etc. are congruent to each other and thus we cannot prove that the perpendicular bisectors are concurrent by an argument similar to the one we used when reasoning about regular polygons.



Therefore, every regular polygon is cyclic and thus can be circumscribed by a circle. While every regular polygon is cyclic, it is not true that every cyclic polygon is regular. We have already seen that every triangle is cyclic, but every triangle is not an equilateral triangle and thus not regular. So it is possible to have polygons that are not regular, but are cyclic as shown in the figure below.

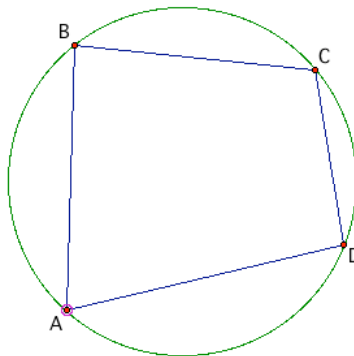


Mathematical Focus 3

A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.

We can establish that a quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary. Proving that two conditions (a quadrilateral being cyclic and its opposite angles being supplementary) are equivalent requires proving an implication and its converse. That is, to prove $A \Leftrightarrow B$ we need to prove $A \Rightarrow B$ and $B \Rightarrow A$.

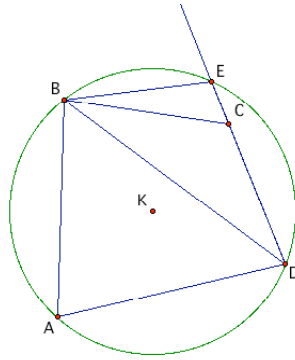
First we prove that given a quadrilateral that can be inscribed in a circle, its opposite angles are supplementary. In the cyclic quadrilateral ABCD shown in the following diagram, $\angle ABC$ is opposite $\angle CDA$ and arcs CDA and ABC form the entire circle. Since the measure of an inscribed angle is half of the measure of the arc in which it is inscribed, the sum of the measures of the two angles must be 180. Thus $\angle ABC$ and $\angle CDA$ are supplementary.



Quadrilateral ABCD

Next we prove that if the opposite angles of a quadrilateral are supplementary, then the quadrilateral can be inscribed in a circle. First, construct circumcircle K for triangle ABD, and then construct quadrilateral ABCD, such that $m\angle BAD + m\angle BCD = 180$. By showing that it is impossible for point C to be either in the interior or the exterior of circumcircle K, it can be concluded that point C must be on circumcircle K and hence, quadrilateral ABCD can be inscribed in a circle.

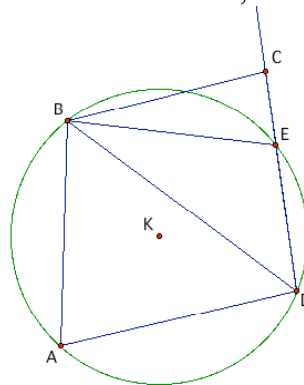
Suppose point C is in the interior of the circle, as shown in the diagram below.



Quadrilaterals ABCD and ABED with C in the interior of the circle

If point C is in the interior of the circle, then point E would be the intersection of \overline{DC} with circle K. Then quadrilateral ABED would be inscribed in circle K and $\angle BED$ would also be supplementary to $\angle BAD$. But then both $\angle BED$ and $\angle BCD$ would be supplementary to $\angle BAD$, so $\angle BED$ would be congruent to $\angle BCD$, implying that \overline{BC} would be parallel to \overline{BE} , which is impossible. Hence, point C cannot be in the interior of circle K.

Now suppose point C is in the exterior of the circle, as shown in the diagram below.



Quadrilaterals ABCD and ABED with C in the exterior of the circle

By the same argument, \overline{BC} would have to be parallel to \overline{BE} , which is impossible. Thus point C must be on circle K, and quadrilateral ABCD is inscribed in circle K.

A corollary that is implied by the above result is that every rectangle is cyclic. Also, no parallelograms other than rectangles are cyclic. Moreover, every isosceles trapezoid is cyclic. [Note: A commonly used definition of trapezoid is that it is a quadrilateral with exactly one pair of parallel sides. However, trapezoids are defined by some sources as a quadrilateral with at least one set of parallel sides. If trapezoids are defined this way, then every rectangle is an isosceles trapezoid.]

Post-Commentary

In making mathematical statements, it is important to recognize which are biconditional and which are not. In Focus 3, we proved a statement and its converse in order to prove the biconditional. In Focus 2, the converse of the statement we proved is not true and so the biconditional is not true. In Focus 1, the use of the biconditional was not as overt. In that Focus, we used the property that each point on a perpendicular bisector of a segment is equidistant from the endpoints of the segment. Later in the focus, in order to establish uniqueness, we needed the result that points that are equidistant from the endpoints of a segment lie on the perpendicular bisector of the segment. Since this statement was the converse of the earlier property, it requires a proof.