

## MAC-CPTM Situations Project

### Situation 45: Zero-Product Property

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#### **Prompt**

A student in Algebra I class wrote the following solution on a homework problem:

$$\begin{aligned}x^2 - 4x - 5 &= 7 \\(x - 5)(x + 1) &= 7 \\x - 5 = 7 \quad x + 1 = 7 \\x = 12 \quad x = 6\end{aligned}$$

A different student commented that 6 indeed was solution to the equation since  $6^2 - 4(6) - 5 = 7$ , but that 12 was not.

#### **Commentary**

In this problem, the student uses an overgeneralization of the Zero Product property. The Zero Product Property states that if the product of two quantities  $ab = 0$ , then it must be the case that  $a = 0$  or  $b = 0$ . This can be justified by making the comment that any number times zero equals zero. There is no other number that is discussed in school mathematics where this property will always hold. In this problem, the student extends this property to products of numbers other than zero. The inaccuracy of this extension will be discussed in Focus 1. When discussing quadratic polynomials like these, factors are a recurring theme. In Focus 2, the factors of quadratic polynomials are discussed in greater detail, paying particular attention to the connection between the roots of the polynomial and the factors of the polynomial. If thought of more abstractly, polynomials with integer coefficients form the object known as a ring in Abstract Algebra. More specifically is that not only do these polynomials form a ring, but they form the more specific object known as an integral domain. Integral domains and their application to this problem are discussed in Focus 3.

#### **Mathematical Foci**

### **Mathematical Focus 1**

*A well-selected counterexample will disprove that this Seven Product property could not be employed in the same manner as the Zero Product property.*

In stating the Zero Product Property, it is noted that if  $ab = 0$ , then  $a = 0$  or  $b = 0$ . In the solution from above, the student has seemed to create a Seven Product Property, which would imply that if  $ab = 7$ , then  $a = 7$  or  $b = 7$ . However, it is not necessarily the case that if the product of two numbers is seven, then one of those two numbers must be seven. A counterexample that would disprove this Seven Product Property would be  $2 \times 3.5 = 7$ , but neither 2 nor 3.5 are equal to 7.

### **Mathematical Focus 2**

*An examination of the products of the coefficients of the quadratic polynomial will determine its proper roots. Additionally, investigating the factors of the coefficient on the opposite side of the equation will yield extraneous calculations compared to the calculations necessary for the Zero Product Property.*

We can look at this problem by discussing the roots of the function. Since the leading coefficient of the quadratic polynomial is one, we are guaranteed that if rational roots do exist for this polynomial, then those roots must be integers. This is a result from the Rational Root Theorem. We also know that for any quadratic polynomial the sum of the roots must equal the constant term and the sum of the roots must equal the coefficient of the linear term. In the student's solution, the product of the roots is 72 and the sum of the roots is 18. However, the corresponding coefficients for this quadratic polynomial are -5 and -4., which provide the necessary contradiction.

We can look at this polynomial and determine the different possibilities for the factors of the right side of the equation. Since any number times zero is zero, we can solve any product of polynomials by setting each individual factor equal to zero. If the polynomial did not equal zero then the algorithm for determining the solutions becomes much more complicated. Instead of setting each factor equal to zero, we would set each factor equal to one of the factors of the number on the right side of the equation. Given that the solutions must be integers and the number on the right side of the equation is prime, there are only a few possibilities to check. The illustration for this particular problem is given below:

$$x^2 - 4x - 5 = 7$$

$$(x - 5)(x + 1) = 7$$

$$x - 5 = 7, x + 1 = 1 \quad \text{or} \quad x - 5 = 1, x + 1 = 7$$

$$x - 5 = -7, x + 1 = -1 \quad \text{or} \quad x - 5 = -1, x + 1 = -7$$

When given a composite number for the opposite side of the equation, the number of possibilities becomes greater. If we change the 7 to 16, we have many more possible solution paths to investigate:

$$(x - 5)(x + 1) = 16$$

$$x - 5 = 1, x + 1 = 16 \quad \text{or} \quad x - 5 = 16, x + 1 = 1$$

$$x - 5 = -1, x + 1 = -16 \quad \text{or} \quad x - 5 = -16, x + 1 = -1$$

$$x - 5 = 2, x + 1 = 8 \quad \text{or} \quad x - 5 = 8, x + 1 = 2$$

$$x - 5 = -2, x + 1 = -8 \quad \text{or} \quad x - 5 = -8, x + 1 = -2$$

$$x - 5 = 4, x + 1 = 4$$

$$x - 5 = -4, x + 1 = -4$$

In these possibilities from above, the solutions to the corresponding quadratic polynomial  $x^2 - 4x - 21 = 0$  are found by using two of the factors from above:  $x - 5 = 2$  and  $x + 1 = -2$ . It is not obvious that these two factors are the appropriate factors to use to solve the quadratic polynomial correctly. By setting each factor equal to zero, there are fewer equations to solve.

### **Mathematical Focus 3**

*By recognizing that the set of polynomials forms an integral domain, it is possible to use the fact that if the product of two polynomials is zero then one factor must be zero.*

Concepts from abstract algebra could be used to provide an explanation into correctly solving this polynomial. Using the concept of rings, the set of polynomials forms a ring under the standard operations of multiplication and addition. Additionally, the set of polynomials forms an integral domain, which implies that there are no zero divisors that are polynomials. Specifically, this means if the product of two polynomials equals zero, then it must be the case that at least one of the two polynomials is equal to zero. By manipulating the original equation  $x^2 - 4x - 5 = 7$  to an equation set equal to zero  $x^2 - 4x - 12 = 0$  and then to the product of two polynomials equal to zero  $(x - 6)(x + 2) = 0$ , it has been shown that the product of two polynomials equals zero. Since there are no zero divisors, then it is assumed that at least one of these two polynomials must be equal to zero. Therefore, the product of the two polynomials equal to zero can be expressed as setting each individual polynomial equal to zero.

### **Post Commentary**

In his book, Barbeau (2000) proposes a quadratic equation in which it is possible to determine the correct solution by factoring and not using the zero product property. His original problem and solution are given below (p. 17):

$$4 = 6 - x - x^2$$

$$4 = (3 + x)(2 - x)$$

$$4 = 3 + x \quad 4 = 2 - x$$

$$x = 1 \quad x = -2$$

Whereas had the problem been solved correctly, we would get the following:

$$4 = 6 - x - x^2$$

$$0 = 2 - x - x^2$$

$$0 = (2 + x)(1 - x)$$

$$0 = 2 + x \quad 0 = 1 - x$$

$$-2 = x \quad 1 = x$$

Therefore it is possible to determine correctly some of the solutions by factoring and not using the zero product property, but it is an algorithm that cannot be used all the time in all situations.

## References

Barbeau, E.J. (2000). *Mathematical Fallacies, Flaws, and Flimflam*. Washington, DC: Mathematical Association of America.