Situation: Zero-Product Property

Prepared at Penn State
Mid-Atlantic Center for Mathematics Teaching and Learning
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Prompt

An Algebra 1 class was given the following problem on a quiz.

PJ wrote the following solution on a homework problem:

$$x^2 - 4x - 5 = 7$$
$$(x-5)(x+1) = 7$$
$$x - 5 = 7 \quad x + 1 = 7$$
$$x = 12 \qquad x = 6$$

Explain to PJ why 12 and 6 are incorrect answers.

One student said, "But 6 is a correct answer. $6^2 - 4 \cdot 6 - 7$ *equals* 7."

Commentary

This situation is based upon the observation that it is possible for a correct answer to result from the use of an incorrect process.

There are several correct processes one can use to find the solutions to a quadratics equation. However, the focus of this situation is the zero product property.

If $a \cdot b = 0$, then $a = 0 \text{ or } b = 0$.

Mathematical Foci

Mathematical Focus 1

What are the solutions to the original problem?

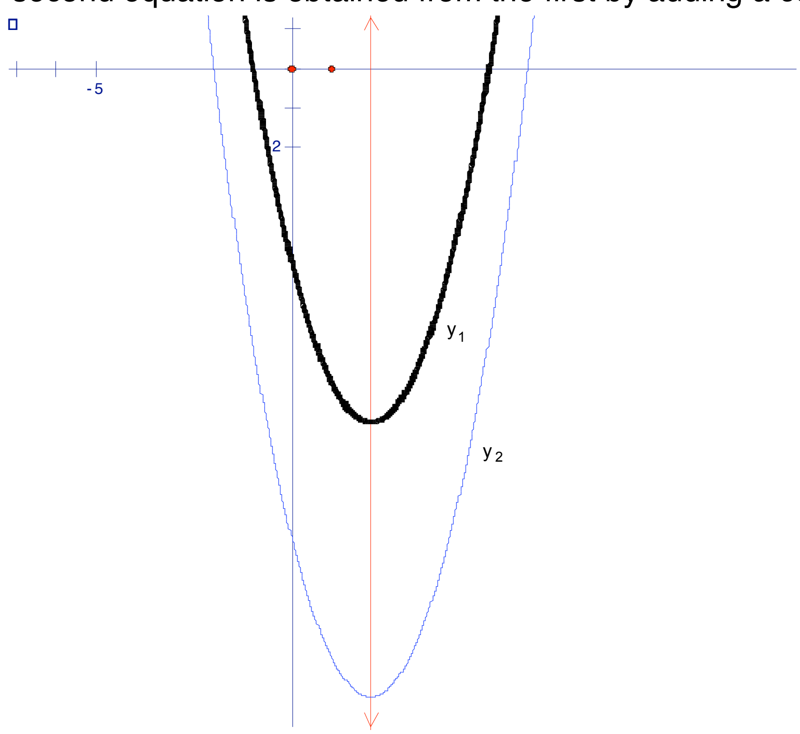
$$x^2 - 4x - 5 = 7$$
$$x^2 - 4x - 12 = 0$$
$$(x-6)(x+2) = 0$$
$$x - 6 = 0 \quad x + 2 = 0$$
$$x = 6 \qquad x = -2$$

Mathematical Focus 2

This prompt illustrates it is possible for at least one solution of a quadratic equation solved incorrectly to be one of the solutions to the same quadratic equation when solved correctly. That is, using the example in the prompt, $x^2 - 4x - 5 = 7$ solved incorrectly has solutions 6 and 12 and solved correctly has solutions 6 and -2. The value, 6, is common to both. The question addressed in this focus: **under what conditions does this occur?**

Observation #1: $x^2 - 4x - 5 = 7$ and $x^2 - 4x - 12 = 0$ are equivalent equations.

Observation #2: The graphs of functions related to these equations, $y = x^2 - 4x - 5$ and $y = x^2 - 4x - 12$, have the same axis of symmetry, x = 2 and differ by a vertical translation. (See Fig. 1.) This observation makes sense because the second equation is obtained from the first by adding a constant.



$y_1 = x^2 - 4x - 5$ and $y_2 = x^2 - 4x - 12$

Figure 1

Observation #3: The left x-intercepts of the parabolas are 1 unit apart, and the right x-intercepts of the parabolas are 1 unit apart.

Represent the equivalent equations more generally, as $x^2 + bx + c = d$ and $x^2 + bx + (c - d) = 0$, respectively. The parabolas given by $y_1 = x^2 + bx + c$ and $y_2 = x^2 + bx + (c - d)$ will have the same axis of symmetry, $x = -b/2$. What remains to be shown is that when $d \neq 0$ the left x-intercepts of $y_1 = x^2 + bx + c$ and $y_2 = x^2 + bx + (c - d)$ are 1 unit apart and the right x-intercepts of $y_1 = x^2 + bx + c$ and $y_2 = x^2 + bx + (c - d)$ are 1 unit apart.

First, we'll show how to generate problems for which

Consider the equation $x^2 + bx + c = d$.

Using the quadratic formula,

$$x^2 + bx + c = \left(x - \frac{-b + \sqrt{b^2 - 4c}}{2}\right)\left(x - \frac{-b - \sqrt{b^2 - 4c}}{2}\right) = d$$

Using PJ's incorrect method,

$$x - \frac{-b + \sqrt{b^2 - 4c}}{2} = d \qquad \text{or} \qquad x - \frac{-b - \sqrt{b^2 - 4c}}{2} = d$$

$$x = d + \frac{-b + \sqrt{b^2 - 4c}}{2} \qquad\qquad x = d + \frac{-b - \sqrt{b^2 - 4c}}{2}$$

Therefore, using PJ's incorrect method, the solutions of $x^2 + bx + c = d$ are $x = d + \frac{-b \pm \sqrt{b^2 - 4c}}{2}$.

Now consider $x^2 + bx + (c - d) = 0$.

Using the quadratic formula, the solutions of $x^2 + bx + (c - d) = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4(c - d)}}{2}$.

Next, determine the values of d for which the solutions are equal.

$$d + \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{-b \pm \sqrt{b^2 - 4(c-d)}}{2}$$
$$2d - b \pm \sqrt{b^2 - 4c} = -b \pm \sqrt{b^2 - 4c + 4d}$$
$$2d \pm \sqrt{b^2 - 4c} = \pm\sqrt{b^2 - 4c + 4d}$$
$$\left(2d \pm \sqrt{b^2 - 4c}\right)^2 = \left(\pm\sqrt{b^2 - 4c + 4d}\right)^2$$
$$4d^2 \pm 4d\sqrt{b^2 - 4c} + b^2 - 4c = b^2 - 4c + 4d$$
$$4d^2 \pm 4d\sqrt{b^2 - 4c} = 4d$$
$$4d^2 \pm 4d\sqrt{b^2 - 4c} - 4d = 0$$
$$4d\left(d \pm \sqrt{b^2 - 4c} - 1\right) = 0$$

Either $d = 0$ or $d = 1 \pm \sqrt{b^2 - 4c}$.

So additional examples like the one PJ encountered are ones for which $d = 0$ or $d = 1 \pm \sqrt{b^2 - 4c}$. Convenient examples are ones for which $b^2 - 4c$ is a perfect square. For example, if b = 12 and c = 27, then $b^2 - 4c$ = 36 and $d = 1 \pm 6$. So d = 7 or d = -5. Then, an example like PJ's would arise from $x^2 + 12x + 27 = 7$ and from $x^2 + 12x + 27 = -5$. In the first case, the solutions to $x^2 + 12x + 27 = 7$ are -10 and -2 and the "solutions" using PJ's incorrect method are –2 and 4. In the second case, the solutions to $x^2 + 12x + 27 = 7$ are -8 and -4 and the "solutions" using PJ's incorrect method are -14 and -8.

Now we'll show that the x-intercepts of the two parabolas generated from two equations are separated by 1 in cases like the ones PJ on which used his incorrect method.

Now use the values of d to determine the x- intercepts of $y_1 = x^2 + bx + c$ and $y_2 = x^2 + bx + (c - d)$.

Substituting $d = 0$ into y_2 results in $y_2 = x^2 + bx + c$. The x-intercepts for y_1 and y_2 are $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$. However, this case does not address the spirit of PJ's question.

Substituting $d = 1 \pm \sqrt{b^2 - 4c}$ into y_2 results in $y_2 = x^2 + bx + \left(c - 1 \pm \sqrt{b^2 - 4c}\right)$.

The x-intercepts for y_1 are $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$.

The x-intercepts for y_2 are $x = \dfrac{-b \pm \sqrt{b^2 - 4\left(c - 1 \pm \sqrt{b^2 - 4c}\right)}}{2}$.

Now, write the x-intercepts for y_2 in terms of the x-intercepts for y_1.

$$x = \frac{-b \pm \sqrt{b^2 - 4\left(c - 1 \pm \sqrt{b^2 - 4c}\right)}}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4c + 4 \pm 4\sqrt{b^2 - 4c}}}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4c \pm 4\sqrt{b^2 - 4c} + 4}}{2}$$

$$x = \frac{-b \pm \sqrt{\left(\sqrt{b^2 - 4c} \pm 2\right)^2}}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4c} \pm 2}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \pm \frac{2}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \pm 1$$

Hence, for any two parabolas of the form $y_1 = x^2 + bx + c$ and $y_2 = x^2 + bx + (c - d)$ such that $d \neq 0$, the x-intercepts are 1 unit apart and the y-intercepts are 1 unit apart.

Mathematical Focus 3

Generalize to two equations of the form $ax^2 + bx + c = d$ and $ax^2 + bx + (c - d) = 0$.

Determine the conditions for which at least one solution solved using the incorrect process described in the vignette to be one of the solutions to the same quadratic equation when solved correctly.

Consider the equation $ax^2 + bx + c = d$.

Using the quadratic formula, the factors of $ax^2 + bx + c$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

As a result, $ax^2 + bx + c = a\left(x - \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = d$

Using PJ's incorrect method,

$$x-\frac{-b+\sqrt{b^2-4ac}}{2a}=d \qquad \text{or} \qquad x-\frac{-b-\sqrt{b^2-4ac}}{2a}=d$$
$$x=d+\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad\qquad x=d+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Therefore, using the incorrect method, the solutions of $ax^2+bx+c=d$ are
$x=d+\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Now consider $ax^2+bx+(c-d)=0$.

Using the quadratic formula, the solutions of $ax^2+bx+(c-d)=0$ are
$x=\frac{-b\pm\sqrt{b^2-4a(c-d)}}{2a}$.

Next, determine the values of d for which the solutions are equal.

$$\begin{aligned}
d+\frac{-b\pm\sqrt{b^2-4ac}}{2a} &= \frac{-b\pm\sqrt{b^2-4a(c-d)}}{2a}\\
2ad-b\pm\sqrt{b^2-4ac} &= -b\pm\sqrt{b^2-4ac+4ad}\\
2ad\pm\sqrt{b^2-4ac} &= \pm\sqrt{b^2-4ac+4ad}\\
\left(2ad\pm\sqrt{b^2-4ac}\right)^2 &= \left(\pm\sqrt{b^2-4ac+4ad}\right)^2\\
4a^2d^2\pm4ad\sqrt{b^2-4ac}+b^2-4ac &= b^2-4ac+4ad\\
4a^2d^2\pm4ad\sqrt{b^2-4ac} &= 4ad\\
4a^2d^2\pm4ad\sqrt{b^2-4ac}-4ad &= 0\\
4ad\left(ad\pm\sqrt{b^2-4ac}-1\right) &= 0
\end{aligned}$$

Either $d=0$ or $d=\frac{1\pm\sqrt{b^2-4ac}}{a}$.

Now use the values of d to determine the x- intercepts of $y_1=ax^2+bx+c$ and $y_2=ax^2+bx+(c-d)$.

Substituting $d=0$ into y_2 results in $y_1=ax^2+bx+c$. The x-intercepts for y_1 and y_2 are $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. However, this case does not address the spirit of PJ's question.

Substituting $d = \frac{1 \pm \sqrt{b^2 - 4ac}}{a}$ into y_2 results in $y_2 = ax^2 + bx + \left(c - \frac{1 \pm \sqrt{b^2 - 4ac}}{a}\right)$.

The x-intercepts for y_1 are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The x-intercepts for y_2 are $x = \frac{-b \pm \sqrt{b^2 - 4a\left(c - \frac{1 \pm \sqrt{b^2 - 4ac}}{a}\right)}}{2a}$.

Now, write the x-intercepts for y_2 in terms of the x-intercepts for y_1.

$$x = \frac{-b \pm \sqrt{b^2 - 4a\left(c - \frac{1 \pm \sqrt{b^2 - 4ac}}{a}\right)}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac + 4 \pm 4\sqrt{b^2 - 4ac}}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac \pm 4\sqrt{b^2 - 4ac} + 4}}{2a}$$

$$x = \frac{-b \pm \sqrt{\left(\sqrt{b^2 - 4ac} \pm 2\right)^2}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac} \pm 2}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \pm \frac{2}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \pm \frac{1}{a}$$

Hence, for any two parabolas of the form $y_1 = ax^2 + bx + c$ and $y_2 = ax^2 + bx + (c - d)$ such that $d \neq 0$, the left x-intercepts are $1/a$ units apart and the right x-intercepts are $1/a$ units apart.

Mathematical Focus 4

This prompt illustrates that it is possible for at least one solution of a quadratic equation solved incorrectly to be one of the solutions to the same quadratic equation when solved correctly. This next focus examines the situation when the incorrect process results in the same two values as when the equation is solved correctly. The question posed (but not yet addressed) in this focus: **under what conditions does this occur?**

From Barbeau, 2000, p. 17—

Solved incorrectly:

$6 - x - x^2 = 4$

$(2 - x)(3 + x) = 4$

$2 - x = 4 \quad 3 + x = 4$

$x = -2 \qquad x = 1$

Solved correctly:

$6 - x - x^2 = 4$

$2 - x - x^2 = 0$

$(2 + x)(1 - x) = 0$

$2 + x = 0 \quad 1 - x = 0$

$x = -2 \qquad x = 1$