MAC-CPTM Situations Project

Situation 45: Zero-Product Property

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Prompt

A student in Algebra I class wrote the following solution to a homework problem:

$$x^2 - 4x - 5 = 7$$
$$(x-5)(x+1) = 7$$
$$x - 5 = 7 \quad x + 1 = 7$$
$$x = 12 \qquad x = 6$$

A different student commented that 6 was a solution to the equation since $6^2 - 4(6) - 5 = 7$, but that 12 was not.

Commentary

In this Prompt, the student uses an overgeneralization of the Zero Product Property. The Zero Product Property states that if $ab = 0$, then $a = 0$ or $b = 0$. The only real number, n, for which the property "If $ab = n$, then $a = n$ or $b = n$" holds for all real values a and b, is $n = 0$. In this Prompt, the student extends the Zero Product Property to values of n other than 0. The inaccuracy of this extension will be discussed in Focus 1. Factors are a recurring theme in the study of quadratic polynomials, so factors are addressed in Foci 2 and 3. In these Foci we consider some special cases in which the student's proposed property and solution method will work. Finally, considering this Situation from an abstract algebra standpoint, the set of polynomials forms an object known as an integral domain. Integral domains and their application to this problem are discussed in Focus 4.

Mathematical Foci

Mathematical Focus 1

A well-chosen counterexample will prove that a "Seven Product Property" may not be employed in the same manner as the Zero Product Property.

The Zero Product Property states that if $ab = 0$, then $a = 0$ or $b = 0$. In the solution given in the Prompt, the student seems to have created a "Seven Product Property," which would imply that if $ab = 7$, then $a = 7$ or $b = 7$. However, it is not necessarily the case that if the product of two numbers is 7, then one of the numbers must be 7. One counterexample that would disprove this "Seven Product Property" is $2 \times 3.5 = 7$; the product is 7, but neither of the factors is 7.

The converse of the Zero Product Property is also true: If $a = 0$ or $b = 0$, then $ab = 0$. So another valuable counterexample to the "Seven Product Property" comes from an investigation of its converse: If $a = 7$ or $b = 7$, then $ab = 7$. If one of the factors is 7, and the other factor is not the multiplicative identity (to be discussed in Focus 2), then the product cannot be 7. For example, let $a = 7$ and $b = 10$. The product is 70, not 7, thus disproving the "Seven Product Property."

Mathematical Focus 2

A "Seven Product Property" holds true if and only if one of the factors is the multiplicative identity.

It is important to note the reason that allowed one of the student's answers to be a correct solution even though an incorrect assumption was used. The "Seven Product Property" works under certain conditions. For example, when multiplying two factors, if one of the factors is 7 and the other is the multiplicative identity, 1, then the product is 7. Consider again the student's solution.

$$
\begin{array}{ll}
x^2 - 4x - 5 = 7 & \\
(x-5)(x+1) = 7 & \\
x - 5 = 7 & x + 1 = 7 \\
x = 12 & x = 6
\end{array}
$$

The solution $x = 6$ is correct because $x = 6$ allows one of the factors to equal 7 and the other to equal 1. That is, if $x = 6$, then $(x - 5)(x + 1) = (6 - 5)(6 + 1) = (1)(7) = 7$. The solution $x = 12$ is not correct, however, because although it allows one factor to equal 7, it causes the other factor to equal 13. We know the product of 7 and 13 is not 7. It is also important to note that the other correct solution to the quadratic equation is $x = -2$, which allows one factor to be -1 and the other -7. That is, if $x = -2$, then $(x - 5)(x + 1) = (-2 - 5)(-2 + 1) = (-7)(-1) = 7$.

Mathematical Focus 3

Investigating the factors of the term on the right side of the equation will yield both valid and extraneous solutions, highlighting the relevance and necessity of the Zero Product Property.

It is possible to determine solutions to the equation $x^2 - 4x - 5 = 7$ in a manner similar to the student's strategy, but in general, the student's method is too complicated to be useful for most polynomials. If used correctly, the method would work like this: Factor the polynomial on the left side of the equation into two binomial factors (in this case, $(x - 5)$ and $(x + 1)$), as the student did. Then factor the integer on the right side of the equation into a pair of its factors. In this case, since 7 is prime, the task is not difficult. Now set each factor on the left equal to one of the factors on the right side of the equation and check each possibility to see which one gives correct solutions. The illustration for this particular problem is given below:

$$x^2 - 4x - 5 = 7$$
$$(x - 5)(x + 1) = 7$$

1a) $(x - 5)(x + 1) = (7)(1)$
$x - 5 = 7$ and $x + 1 = 1$
$\boldsymbol{x = 12, 0}$

or

1b) $x - 5 = 1$ and $x + 1 = 7$
$\boldsymbol{x = 6}$

or

2a) $(x - 5)(x + 1) = (-7)(-1)$
$x - 5 = -7$ and $x + 1 = -1$
$\boldsymbol{x = -2}$

or

2b) $x - 5 = -1$ and $x + 1 = -7$
$\boldsymbol{x = 4, -8}$

In the case of 1a) and 2b) above, x would have to take on two different values at once in order for the equation to hold. Since this is impossible, those values for x are not solutions. However, the solutions for 1b) and 2a) are valid. This can be verified by substituting $x = 6$ and $x = -2$ into the original equation:

$$(6)^2 - 4(6) - 5 = 7 \quad \text{and} \quad (-2)^2 - 4(-2) - 5 = 7$$

This method was relatively manageable in this case because 7 is prime. But one can imagine the lengthy process that would be required to check all possible solutions if the integer on the right side of the equation had several factors. For example, if we had a 16 instead of a 7, there are many more possible solutions to investigate:

$$(x-5)(x+1)=16$$

$$x-5=1,\ x+1=16 \quad \text{or} \quad x=5=16,\ x+1=1$$

$$x-5=-1,\ x+1=-16 \quad \text{or} \quad x=5=-16,\ x+1=-1$$

$$x-5=2,\ x+1=8 \quad \text{or} \quad x-5=8,\ x+1=2$$

$$x-5=-2,\ x+1=-8 \quad \text{or} \quad x-5=-8,\ x+1=-2$$

$$x-5=4,\ x+1=4$$

$$x-5=-4,\ x+1=-4$$

The solutions in this case are found by using two of the factors from above: $x - 5 = 2$ and $x + 1 = -2$. However, it is not obvious that these are the appropriate factors to use to solve the quadratic polynomial correctly. This highlights the relevance of employing the Zero Product Property to solve polynomial equations. Because $ab = 0$ implies $a = 0$ or $b = 0$, we have far fewer solutions to check after we factor a particular polynomial. We can simply set each factor equal to zero, resulting in fewer equations to solve. In the case of the equation $x^2 - 4x - 5 = 7$, the solution can be found by first setting the quadratic polynomial equal to zero ($x^2 - 4x - 12 = 0$) and *then* factoring: $(x - 6)(x + 2) = 0$. Since this implies $x - 6 = 0$ or $x + 2 = 0$, the solutions are $x = 6$ and $x = -2$.

Mathematical Focus 4

The Zero Product Property is unique to particular contexts, one of which is the set of polynomials.

The Zero Product Property may seem obvious, but it is unique to particular sets of numbers. The Zero Product Property is relevant in this Situation because (in abstract algebra terms) the set of polynomials is an integral domain and therefore contains no zero divisors. Zero divisors are nonzero elements of a set (we could call the elements a and b), such that $ab = 0$. For example, sets such as the integers, the real numbers, or (in the case of this Situation) the polynomials, have no zero divisors, which is why the Zero Product Property holds for these sets.

This fact may be appreciated by noting some cases in which the Zero Product Property does *not* hold. For example, consider the set Z_6, which consists only of the integers 0, 1, 2, 3, 4, and 5. According to the Zero Product Property, $ab = 0$ implies $a = 0$ or $b = 0$. However, in Z_6, it is possible to construct a product of 0 without either of the factors being 0. The product (2)(3) is such an example. We know that if we are working in the set of all integers, (2)(3) = 6. But since we are in Z_6, (2)(3) = 0. That is, 2 and 3 are zero divisors in Z_6.