MAC-CPTM Situations Project

Situation 46: Division Involving Zero

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Prompt

On the first day of class, preservice middle school teachers were asked to evaluate $\frac{2}{0}$, $\frac{0}{0}$, and $\frac{0}{2}$ and to explain their answers. There was some disagreement among their answers for $\frac{0}{0}$ and quite a bit of disagreement among their explanations:

- Because any number over 0 is undefined;
- Because you cannot divide by 0;
- Because 0 cannot be in the denominator;
- Because 0 divided by anything is 0; and
- Because a number divided by itself is 1.

Commentary

The mathematical issue centers on the possible values that result when zero is the dividend, the divisor, or both the dividend and the divisor in an expression involving division. If zero appears only as the dividend, the expression has a unique value, namely zero. If zero appears only as the divisor, the value of the expression is not defined. If zero appears in both the dividend and the divisor, the value is indeterminate. An important distinction is not being able to determine an answer (e.g., for $\frac{0}{0}$) as opposed to not being able to specify a particular answer (e.g., for $\frac{2}{0}$). The foci use multiple contexts to illustrate the three possible types of results. Connections are made to indeterminate forms, limits, and the real projective line.

Mathematical Focus 1:
An expression involving division can be viewed as a number, so an equation can be written that uses a variable to represent that number. Examining the

number of solutions for equations that are equivalent to that equation can show whether the expression has one value, is undefined, or is indeterminate.

Unique Solution

If $\frac{0}{2} = x$, then $2x = 0$. The unique solution to this equation is $x = 0$.

Infinite Solutions

If $\frac{0}{0} = x$, then $0x = 0$. Because any value of *x* is a solution to this equation, there are an infinite number of solutions, hence, no unique solution.

No Solution

If $\frac{2}{0} = x$, then $0x = 2$. No real number *x* is a solution to this equation.

Mathematical Focus 2

Division can be modeled in two ways, partitive (given the total and the number of groups, determine the number in each group) and quotitive (given the total and the number in each group, determine the number of groups). For each of these, one can find the value of whole-number division expressions by finding either the number of objects in a group or the number of groups.

PARTITIVE DIVISION

Unique Solution

$\frac{12}{3}$ can be modeled by sharing 12 objects equally among 3 groups and asking how many objects would be in one group. $\frac{12}{3}$ could also be modeled using portions, for example: “If 12 is 3 portions, how many is 1 portion?” So $\frac{0}{2}$ can be thought of as 0 objects in 2 groups, which means 0 objects per 1 group, or 0 objects comprising 1 portion.

Infinite Solutions

$\frac{0}{0}$ can be modeled using portions, for example: “If 0 is 0 portions, how many is 1 portion?” There is not enough information to answer the question. If a portion is 3, or 7.2, or any size at all, 0 portions would be 0.

No Solutions

Similarly, $\frac{2}{0}$ can be modeled using portions, for example: “If 2 is 0 portions, how many is 1 portion?” In this case the portion size is undefined, because there are 0 portions.

QUOTATIVE DIVISION

Unique Solution

$\frac{12}{3}$ can be modeled by splitting 12 objects into groups of 3, or alternatively as 12 split into portions of size 3, and asking how many groups or portions can be made. So $\frac{0}{2}$ can be thought of as splitting 0 objects in groups of 2, which means 0 groups, or 0 portions of size 2.

Infinite Solutions

$\frac{0}{0}$ can be modeled by splitting 0 objects into groups of 0, and asking how many groups can be made. Since there could be any number of groups, there are an infinite number of solutions.

No Solution

$\frac{2}{0}$ can be modeled by splitting 2 objects into groups of 0, and asking how many groups can be made. Regardless of how many groups of 0 we remove, no objects are removed. Therefore, the number of groups is undefined.

Mathematical Focus 3:

Different mathematical representations can be used to

Slope

The slope of a line between two points in the Cartesian plane can be defined as the ratio of the change in the y-direction to the change in the x-direction, or as the rise divided by run.

Infinite Solutions

In the case of two coincident points, the change in the y-direction and the change in the x-direction are both 0, which means that the rise divided by run is $\frac{0}{0}$. There are an infinite number of lines through two coincident points.

No Solution

In the case of two points lying on the same vertical line whose y-coordinates differ by a, the change in the y direction will be a and the change in the x direction is 0. Since $\frac{rise}{run}$ is $\frac{a}{0}$, the slope of a vertical line is undefined.

Direct Proportion

Suppose $y = kx$ is a direct proportion. For points on the line, the ratio $\frac{y}{x}$ equals k, which is constant.

Infinite Solutions
Because the origin is on the line $y = kx$, it appears that $\frac{0}{0} = k$. But, because this would work for any constant of proportionality, $\frac{0}{0}$ could be any number based on similar reasoning. So $\frac{0}{0}$ is undefined.

No Solution
The case of $\frac{2}{0}$ is difficult to explain via the language of direct proportion, but the graph would be a vertical line, and there is a sense in which $k = \infty$, as will be seen subsequently.

Cartesian Product

Unique Solution
If 12 outfits can be made using 3 pairs of pants and some number of shirts, how many shirts are there? Similarly, if 0 outfits can be made using 2 pairs of pants and some number of shirts, there must be 0 shirts.

Infinite Solutions
If 0 outfits can be made using 0 pairs of pants and some number of shirts, the number of possibilities for the number of shirts is infinite.

No Solution
How shirts are there if there are two outfits and 0 pairs of pants? No possible number of shirts can be used to make 2 outfits if there are 0 pairs of pants.

Factor Pairs

Unique Solution
For $\frac{12}{3}$, 3 and the quotient are a factor pair for 12 (and 12 is a multiple of 3). For $\frac{0}{2}$, 2 and the quotient are a factor pair for 0. Therefore, the quotient must be 0.

Infinite Solutions

For $\frac{0}{0}$, 0 is part of an infinite number of factor pairs for 0 (and 0 is a multiple of 0).

No Solution

For $\frac{2}{0}$, 0 is not part of any factor pair for 2 (and 2 is not a multiple of 0).

Mathematical Focus 4:

Division involving zero can be examined via contextual applications of division.

Speed

Unique Solution

If one goes 12 miles in 3 hours, how fast is one going? The answer is 4 miles per hour, and the answer is produced using division. If one goes 0 miles in 2 hours, one is going 0 miles per hour.

Infinite Solutions

If one goes 0 miles in 0 hours, how fast is one going? An infinite number of speeds are possible.

No Solution

If one goes 1 mile in 0 hours, how fast is one going? This is impossible. [Note that there is a sense of infinite speed here.]

Unit Price

Unique Solution

If $12 buys 3 pounds of tomatoes, how much is 1 pound? If $0 buys 2 pounds of tomatoes, then 1 pound can be bought for $0.

Infinite Solutions

If $0 buys 0 pounds of tomatoes, there are an infinite number of possible costs for 1 pound.

No Solution

If $2 buys 0 pounds of tomatoes, no possible number of dollars can buy 1 pound.

Rate

Unique Solution
If Angela makes 3 free throws in 12 attempts, what is her rate? If Angela makes 0 free throws in 2 attempts, her rate is 0.

Infinite Solutions
If Angela makes 0 free throws in 0 attempts, her rate could be any of an infinite number of rates.

No Solution
Since it is not possible for Angela to make 2 free throws in 0 attempts, it is not possible to determine her rate.

Rectangle Area

Suppose we allow that rectangles can have side lengths of 0.

Unique Solution
If a rectangle has area 12 and height 3, what is its width? Answer: 4. If a rectangle has area 0 and length 2, its width is 0.

Infinite Solutions
If a rectangle has area 0 and height 0, what is its width? Any width is possible.

No Solution
If a rectangle has area 2 and height 0, what is its width? It is impossible for a rectangle to have area 2 and height 0. [Note: In fact, this is the Dirac delta function.]

Mathematical Focus 5:

Indeterminate forms can be used to examine the value of expressions such as $\frac{0}{0}$.

The expression $\frac{0}{0}$ is an indeterminate form. When determining the limit, $\frac{0}{0}$ can "become" anything. For example, in the case of $\lim_{x\to 0}\frac{\sin x}{x}$, the limit is 1. However, in the case of $\lim_{x\to 0}\frac{2\sin x}{3x}$, the limit is $\frac{2}{3}$.

In contrast, $\frac{2}{0}$ is not typically called an indeterminate form, and usually the limit does not exist. For example, in the case of $\lim_{x\to 0}\frac{2}{x}$, the limit does not exist because it approaches $-\infty$ from the left and ∞ from the right. Sometimes it is appropriate to say that the limit is infinity. For example, $\lim_{x\to 0}\frac{2}{x^2} = \infty$.

Mathematical Focus 6:
The real projective line

In the Cartesian plane, consider the set of lines through the origin, and consider each line to be an equivalence class of points in the plane.

Except when $x = 0$, the ratio of the coordinates of a point gives the slope of the line that is the equivalence class containing that point. The origin must be excluded because it would be in all equivalence classes, which is rather like saying that $\frac{0}{0}$ would be the slope of any line through the origin. Note that the slope of a line through the origin is equal to the y-coordinate of the intersection of that line and the line $x = 1$. This way, we can use slope to establish a one-to-one correspondence between the equivalence classes and the real numbers. Thus, the real numbers give us all possible slopes, except for the vertical line.

When $x = 0$, all the points in the equivalence class lie on the vertical line that is the y-axis. (Again the origin must be excluded from this equivalence class.) The ratio of the coordinates is undefined, so the slope is undefined. As positively sloped lines approach vertical, their slopes approach ∞, suggesting the slope of the vertical line to be ∞. As negatively sloped lines approach vertical, their slopes approach –∞, suggesting the slope should instead be –∞. However, there is only one vertical line through the origin, so it cannot have two different slopes. To resolve this ambiguity, we can decide that ∞ and –∞ are the same "number" because they should represent the same slope. So now, if we think about all possible slopes, we have all real numbers and one more number, which we will call ∞. Imagine beginning with the extended real line, $\Re \cup \{\infty, -\infty\}$, and gluing together the points ∞ and –∞ so that they are the same point. This is the real projective line, $\Re \cup \{\infty\}$.