MAC-CPTM Situations Project

Situation 47: Graphing Inequalities with Absolute Values

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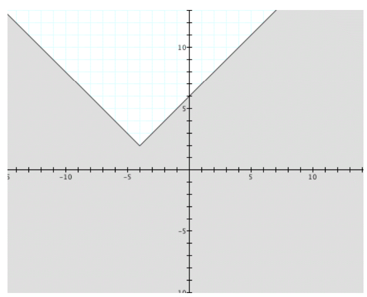
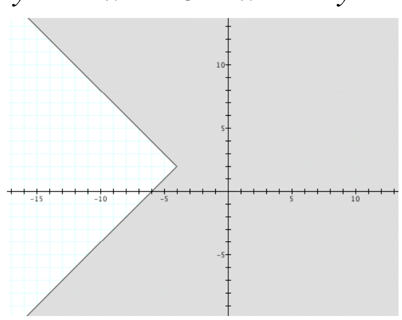
Prompt

This episode occurred during a course for prospective secondary mathematics teachers. The discussion focused on the graph of $y - 2 \le |x + 4|$. The instructor demonstrated how to graph this inequality using compositions of transformations, generating the following graph.

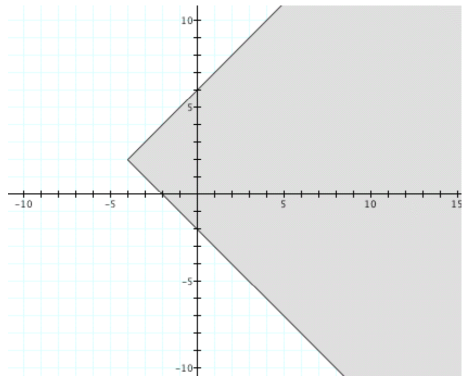


Students proposed other methods, which included the two different algebraic formulations and accompanying graphs as seen below.

$$y - 2 \le x + 4 \text{ or } -x - 4 \le y - 2$$



$$y - 2 \le x + 4 \text{ and } -x - 4 \le y - 2$$



Students expected their graphs to match the instructor's graph, and they were confused by the differences they saw.

Commentary

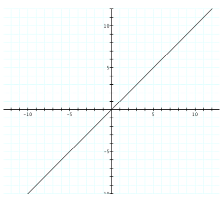
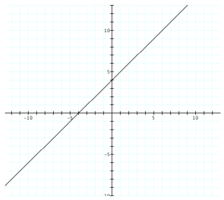
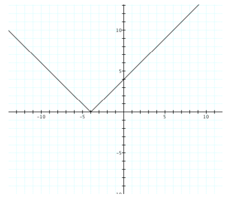
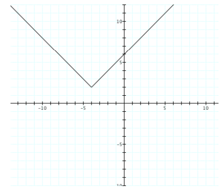
This prompt involves the graph of an absolute value inequality involving two variables. The graph of an absolute value inequality is related to the graph of an absolute value function. Absolute values can be interpreted through a symbolically stated definition of absolute value as well as through the conception of absolute value as distance from zero. Composite functions such as absolute value functions can be thought about as transformations of input and/or output values of functions. Working with absolute values entails keeping track of the logic of the conjunctions and disjunctions into which absolute value statements transform, and working with inequalities involves applying an appropriate range of rules governing their transformation. Inequalities in two variables can be interpreted by examining the inequality for each of a series of constant values for one of the variables.

Mathematical Foci

Mathematical Focus 1

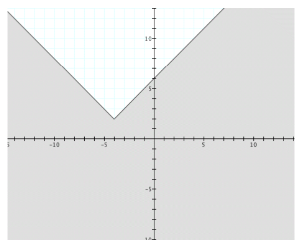
An absolute value inequality that involves a composite function can be graphed by a sequence of graphical representations of component functions or by a sequence of transformations on the input and/or output of the parent absolute value function.

One way to think about graphing an inequality with absolute values is to consider the related function as a composition of functions. To graph $y - 2 = |x + 4|$, one can rewrite it as $y = |x + 4| + 2$, and then view this as a composition of functions, m, where $m = k \circ h \circ g \circ f$ for functions f, g, h, and k with rules $f(x)=x$, $g(x)=x+4$, $h(x)=|x|$, and $k(x)=x+2$. Each successive composition transforms the graph as shown in the following sequence of figures.

$f(x) = x$ $\quad$ $g(f(x)) = x + 4$ $\quad$ $h(g(f(x))) = |x + 4|$ $\quad$ $k(h(g(f(x)))) = |x + 4| + 2$

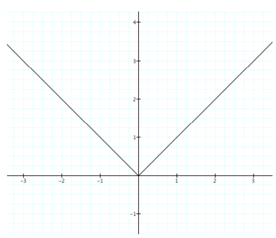
The portion of the plane to be shaded includes all points for which $y - 2$ is less than $|x + 4|$, or, equivalently, all points for which y is less than $|x + 4| + 2$ so the part of the plane below the graph of the function m should be shaded. Alternatively, one could test a point on either side of the graphed function to determine which portion of the plane to shade. A rationale for testing a single point is related to what happens on each vertical line of the shaded graph. Consider a vertical line $x = k$. For some point (x, y) on that line, $y - 2 \leq |x + 4|$ or $y = |x+4|+2$. The y-values above that line will exceed $|k+4|+2$ and the y-values below that line will be less that $|k+4|+2$.



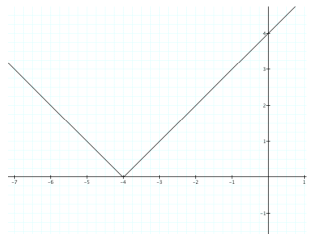
$y - 2 \leq |x + 4|$

This graph is the same as the instructor's graph. The students' graphs are both similar to and different from this graph in two important ways. First, the boundaries of the students' graphs lie on the same lines as the boundaries of this graph. Second, the shading of the students' graphs is to the left or right of (rather than above or below) these boundaries. This combination of correct boundaries and incorrect shading suggests the students may have produced the graph that has the absolute value applied to $y - 2$ and not to $x + 4$. The students' graphs, when analyzed from a transformational perspective, are clearly the graphs of $-|y - 2| \leq x + 4$ and $|y - 2| \leq x + 4$. Thus it seems that the students interpreted the original absolute value inequality in a way that "switched" the expression to which the absolute value applied. The student applied the absolute value to the expression containing y instead of to the expression containing x.

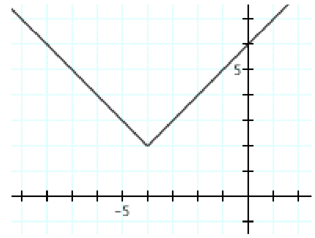
An alternative way to graph the absolute value inequality $y - 2 \leq |x + 4|$ is to start with the graph of the absolute value function and transform it through a sequence of transformations of either the input or the output. The progression of graphs might look like the following.



$y = |x|$

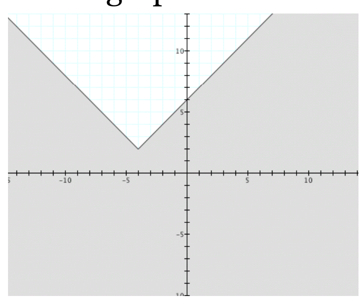


$y = |x + 4|$



$y = |x + 4| + 2$

In this approach, the parent absolute value function is being transformed, with the input value transformed in the first transformation and the resulting output value being transformed in the second transformation. As previously described, after the related function has been graphed, the graph of the inequality requires shading the appropriate side of the graph of the related equation.



$y - 2 \le |x + 4|$

Mathematical Focus 2

A definition of absolute value can be used to translate an absolute value inequality into a statement composed of conjunctions and disjunctions of statements that involve no absolute values.

One way in which absolute value is defined is:

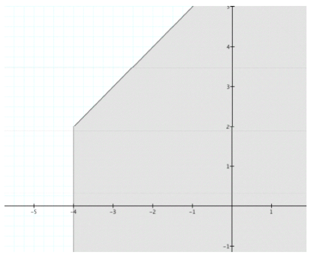
$$|x| = \left\{ \begin{array}{ll} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{array} \right\}.$$

Using this symbolically stated definition of absolute value, the equation $y - 2 = |x + 4|$ can be interpreted using two cases, one for which the input for the absolute value function is positive and one for which the input for the absolute value function is negative. This is usually written as $y - 2 \le \begin{cases} x + 4 \text{ if } x + 4 \ge 0 \\ -(x + 4) \text{ if } x + 4 < 0 \end{cases}$.

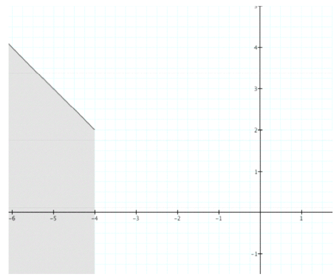
This system of inequalities can be stated using conjunctions and disjunctions as

$$(y - 2 \le x + 4 \quad \text{and} \quad x + 4 \ge 0) \quad \text{or} \quad (y - 2 \le -(x + 4) \quad \text{and} \quad x + 4 < 0).$$

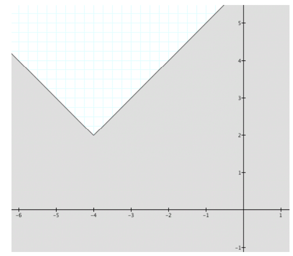
When this system is graphed, it produces the same graph as that produced using the transformational approach to the problem. See the following sequence of graphs.



$y - 2 \le x + 4$
and
$x + 4 \ge 0$



$y - 2 \le -x - 4$
and
$x + 4 < 0$



$(y - 2 \le x + 4 \quad \text{and} \quad x + 4 \ge 0)$
or
$(y - 2 \le -(x + 4) \quad \text{and} \quad x + 4 < 0)$

This system of inequalities is derived directly from the definition of absolute value.

If $a \le |b|$ then $-b \le a \le b$ if $b \ge 0$, and $b \le a \le -b$ if $b < 0$. Equivalently, if $a \le |b|$ then $(a \le b \quad \text{and} \quad -b \le a)$ if $b \ge 0$, and $(a \le -b \quad \text{and} \quad b \le a)$ if $b < 0$. Expressing this system entirely in terms of conjunctions and disjunctions, we can say that if $a \le |b|$ then $\{[(a \le b \quad \text{and} \quad -b \le a) \text{ and } b \ge 0]$, or $[(a \le -b \quad \text{and} \quad b \le a) \text{ and } b < 0]\}$. Graphing $y - 2 \le |x + 4|$ by graphing ($y - 2 \le x + 4$ or $-x - 4 \le y - 2$) suggests a mal-rule of "if $a \le |b|$ then $(a \le b \quad or \quad -b \le a)$ no matter whether b is non-negative or negative. " Graphing $y - 2 \le |x + 4|$ by graphing ($y - 2 \le x + 4$ and $-x - 4 \le y - 2$) suggests a mal-rule or "if $a \le |b|$ then $(a \le b \quad and \quad -b \le a)$ no matter whether b is non-negative or negative."

Mathematical Focus 3

Inequality is not an equivalence relation.

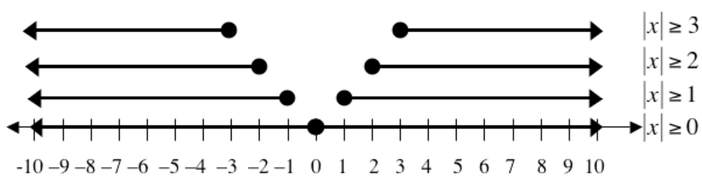
For $a = |b|$, the symmetric property of equality yields equivalent expressions $|b| = a$ and $a = |b|$,which can be written as $-b = a$ and $a = -b$, respectively.

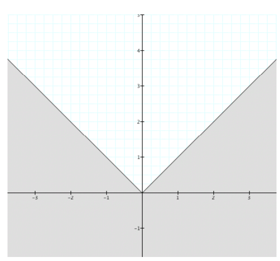
Applying "symmetric property of inequality" would allow one to write $y - 2 \le -(x + 4)$ as $-x - 4 \le y - 2$. However, the inequality relation "$\le$"is not an equivalence relation and fails to satisfy the symmetric property of equivalence relations. Applying to $a \le |b|$ a technique associated with $a = |b|$ without referring to the meaning of the technique is one source of potential error in working with absolute value inequalities.

Mathematical Focus 4

(1) Absolute value can be interpreted as distance from zero.
(2) Inequalities in two variables can be interpreted by examining the inequality for each of a series of constant values for one of the variables.

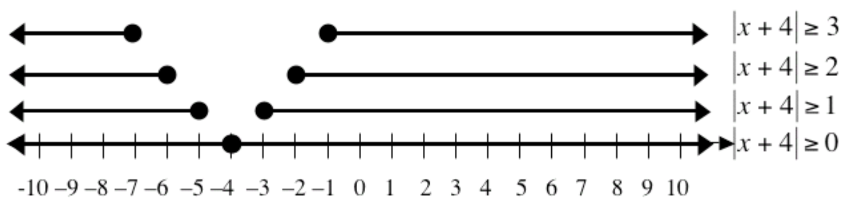
Alternative mathematical definitions can be equivalent, yielding the same results. One interpretation of absolute value is based on distance from zero. We can consider the graph of $y-2 \le |x+4|$ to be all points (x, y) such that $x+4$ is at least $y-2$ units from zero. It is perhaps easier to start by considering the graph of $y \le |x|$ as all points (x, y) such that x is at least y units from zero. We can consider a sequence of rays (pairwise representing the values of x for which $|x| \ge 0$, $|x| \ge 1$, $|x| \ge 2$, $|x| \ge 3$, ...). The rays below represent the pattern we would see for $|x| \ge y$ for non-negative integer values of y. If we imagine a second number line orthogonal to the first (as in the y-axis) and use the fact that $y=|x|$ is continuous, the endpoints of these rays lie on the Cartesian graph of $y=|x|$. One could, in this way, interpret the solutions set of ordered pairs, (x, y), for the inequality, $y \le |x|$, union of the set of all rays $|x| \ge k$ for $k \ge 0$. This strategy of solving equations or inequalities involving two variables by viewing it as the union of a set of equations or inequalities involving only one variable is a case of the common and useful strategy of reducing an unknown problem to a series of problems whose solution strategy is known.

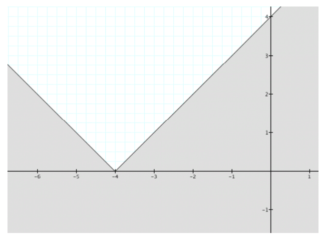




$y \le |x|$

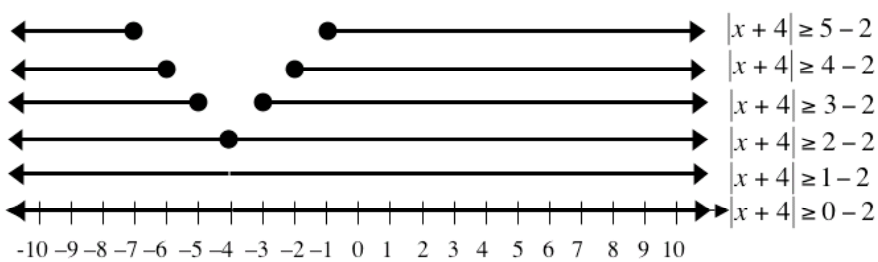
Similarly, consider the graph of $y \le |x+4|$ as all points (x, y) such that $x+4$ is at least y units from zero. We can consider a sequence of rays (pairwise representing the values of x for which $|x+4| \ge 0$, $|x+4| \ge 1$, $|x+4| \ge 2$, $|x+4| \ge 3$, ...) and use the fact that $y=|x+4|$ is continuous and is piecewise defined with two linear functions to form the Cartesian graph of $y=|x+4|$. We can also interpret $y \le |x+4|$ as all points (x, y) such that x is at least y units from -4. In either case, we obtain the sequence of rays shown below.

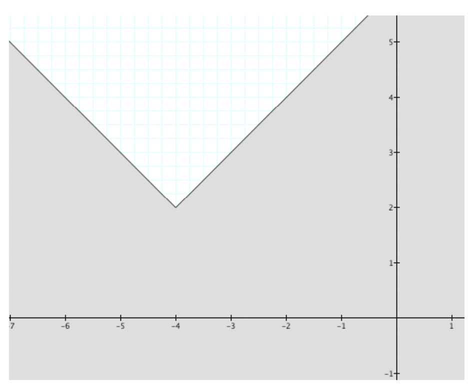




$y \leq |x+4|$

Finally, consider the graph of $y-2 \leq |x+4|$ as all points (x, y) such that $x+4$ is at least $y-2$ units from zero (or, alternatively, all points (x, y) such that *x* is at least $y-2$ units from -4). We can consider a sequence of rays (pairwise representing the values of x for which $|x+4| \geq 0-2,\ |x+4| \geq 1-2,\ |x+4| \geq 2-2,\ |x+4| \geq 3-2,\ \ldots$) and use the fact that $y-2=|x+4|$ is continuous and is piecewise defined with two linear functions to form the Cartesian graph of $y-2=|x+4|$.





$y-2 \leq |x+4|$

Post Commentary

Different definitions afford the opportunity to approach mathematical concepts in substantially different ways. In these foci, viewing absolute value through the lens of a symbolic definition led to a significantly different approach than a view of absolute value as distance.