Situation 48: The Product Rule for Differentiation

Prepared at Penn State
Mid-Atlantic Center for Mathematics Teaching and Learning
060510 – Heather, Shari, Evan

Prompt

In an introductory calculus classroom, a student asks the teacher the following question: "Why isn't the derivative of $y = x^2 \sin x$ just $y' = 2x \cos x$?

Commentary

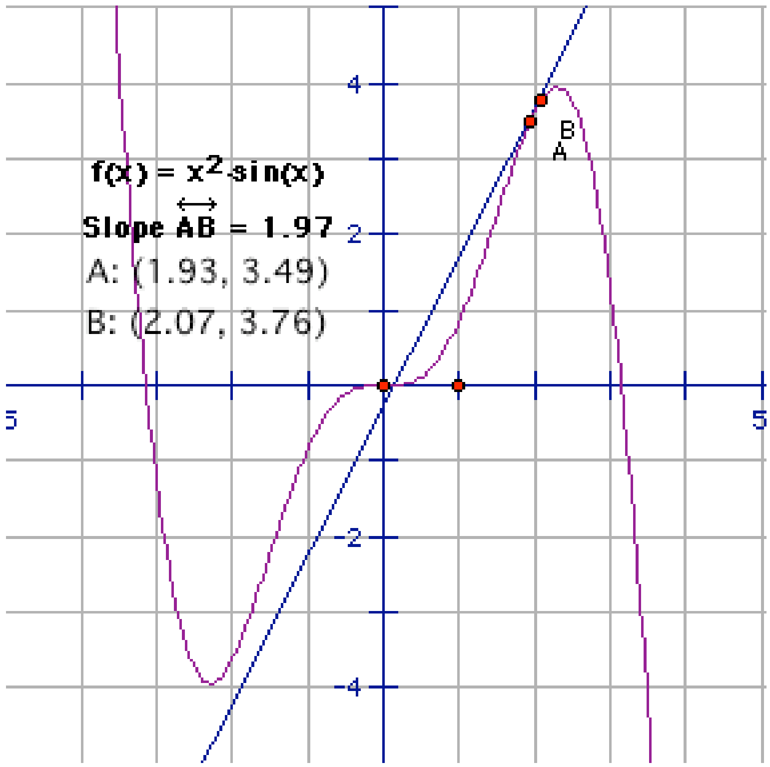
The derivative of a product of functions is not the product of the derivatives of the functions. This situation deals with a common student misconception that the derivative of the product of functions is the product of the derivatives of the functions. The mathematical foci describe different ways of substantiating the product rule for differentiation. The slope of the tangent line is one way to think about the value of the derivative. The definition of the derivative as a limit of a difference quotient is used to generate a symbolic expression for the derivative of the product of two functions (first two specific functions and then two general functions).

Mathematical Foci

Mathematical Focus 1

Derivative as the slope of a tangent line

The derivative of any function represents the slope of the tangent to the graph of the function at every value of x. Therefore, if $y' = 2x \cos x$ is the derivative of $y = x^2 \sin x$, then each value of $y' = 2x \cos x$ will represent the slope of the tangent to $y = x^2 \sin x$ at the corresponding value of x.



Since the x-coordinates of points A and B are extremely close to the value of 2, the slope of the line AB should give a good approximation for the derivative of $y = x^2 \sin x$ at $x = 2$. Therefore, if $y' = 2x\cos x$ is in fact the derivative of $y = x^2 \sin x$, then the value of $y' = 2x\cos x$ at $x = 2$ should be approximately 1.97. Substituting $x = 2$, $y'(2) = 2 \cdot 2\cos(2) \approx -1.665$. This value is not approximately 1.97; therefore $y' = 2x\cos x$ does not represent the slope of the graph of $y = x^2 \sin x$ at every value of x and cannot be its derivative. In addition, by inspection of the graph near the point $x = 2$, one can notice that the function is increasing, not decreasing, therefore a slope of -1.665 would not make sense.

Mathematical Focus 2

Definition of the derivative (specific case)

By definition, the derivative of a function $f(x)$ is given by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Therefore, if $y' = 2x\cos x$ is the derivative of $y = x^2 \sin x$, then it follows that:

$$2x\cos(x) = f'(x) = \lim_{h \to 0} \frac{(x+h)^2 \sin(x+h) - x^2 \sin(x)}{h}.$$

Substituting $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$,

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 \left[\sin(x)\cos(h) + \cos(x)\sin(h)\right] - x^2 \sin(x)}{h}$$

Substituting $\lim_{h \to 0} \cos(h) = 1$,

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2[\sin(x) + \cos(x)\sin(h)] - x^2 \sin(x)}{h}$$

Expanding $(x+h)^2$,

$$f'(x) = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2)[\sin(x) + \cos(x)\sin(h)] - x^2 \sin(x)}{h}$$

Multiplying,

$$f'(x) = \lim_{h \to 0} \frac{\begin{array}{l} x^2 \sin(x) + x^2 \cos(x)\sin(h) + 2xh\sin(x) + 2xh\cos(x)\sin(h) \\ +h^2 \sin(x) + h^2 \cos(x)\sin(h) - x^2 \sin(x) \end{array}}{h}$$

Writing equivalent expressions,

$$f'(x) = \lim_{h \to 0} \left[\frac{x^2 \cos(x)\sin(h)}{h} + \frac{2xh\sin(x)}{h} + \frac{2xh\cos(x)\sin(h)}{h} + \frac{h^2 \sin(x)}{h} + \frac{h^2 \cos(x)\sin(h)}{h} \right]$$

$$f'(x) = \lim_{h \to 0} \left[x^2 \cos(x) \cdot \frac{\sin(h)}{h} + 2x\sin(x) + 2x\cos(x)\sin(h) + h\sin(x) + h\cos(x)\sin(h) \right]$$

Letting $h \to 0$ and recalling that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$,

$$f'(x) = x^2 \cos(x) \cdot 1 + 2x\sin(x) + 0 + 0 + 0$$

Therefore,

$$f'(x) = x^2 \cos(x) + 2x\sin(x).$$

We can show $2x\cos x \neq x^2 \cos(x) + 2x\sin(x)$ if we can evaluate both expressions for a particular value of x, say $x = \pi$ and get different results. When $x=\pi$, we have $2\pi \cos \pi \neq \pi^2 \cos \pi + 2\pi \sin \pi$, since the left side simplifies to 0 and the right side simplifies to 2π. So, $2x\cos x \neq x^2 \cos(x) + 2x\sin(x)$ and $y' = 2x\cos x$ cannot possibly be the derivative of $y = x^2 \sin x$.

Mathematical Focus 3

Definition of the derivative (general case)

The function $k(x) = x^2 \sin x$ is a product of two functions, $f(x) = x^2$ and $g(x) = \sin x$. Therefore, $k(x) = x^2 \sin x$ can be written as $k(x) = f(x) \cdot g(x)$. Using the definition of the derivative, it follows that:

$$k'(x) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

Writing equivalent expressions,

$$k'(x) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x) + f(x+h) \cdot g(x) - f(x) \cdot g(x)}{h}$$

$$k'(x) = \lim_{h \to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$k'(x) = \lim_{h \to 0} \left[f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right]$$

Using the definition of the derivative,

$$k'(x) = \lim_{h \to 0} \left[f(x+h) \cdot g'(x) + g(x) \cdot f'(x) \right]$$

Letting $h \to 0$,

$$k'(x) = \lim_{h \to 0} \left[f(x) \cdot g'(x) + g(x) \cdot f'(x) \right]$$

Therefore,

$$k'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

Since the function $y = x^2 \sin x$ is a product of two functions $f(x) = x^2$ and $g(x) = \sin x$, where $f'(x) = 2x$ and $g'(x) = \cos x$, the derivative of $y = x^2 \sin x$ is given by $y = x^2 \cdot \cos x + (\sin x) \cdot 2x$. Since $2x\cos x \neq x^2 \cdot \cos x + (\sin x) \cdot 2x$, $y' = 2x\cos x$ cannot possibly be the derivative of $y = x^2 \sin x$. We can show $2x\cos x \neq x^2 \cdot \cos x + (\sin x) \cdot 2x$ by showing the left side of the equation has value 0 when x=π and the right side of the equation has value 2π when x=π.

References

Larson, R., Hostetler, R., & Edwards, B. (1999). *Calculus*. New York: Houghton Mifflin.

END OF SITUATION DISCUSSION – END OF SITUATION DISCUSSION