MAC-CPTM Situations Project

Situation 48: The Product Rule for Differentiation

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Prompt

In an introductory calculus classroom, a student asks the teacher the following question:

"Why isn't the derivative of $y = x^2 \sin x$ just $y' = 2x \cos x$?

Commentary

This situation deals with a common student misconception that the derivative of the product of functions is the product of the derivatives of the functions. The mathematical foci describe different ways of challenging this misconception and substantiating the true derivative of a product of functions. The slope of the tangent line is one way to think about the value of the derivative. The definition of the derivative as a limit of a difference quotient is used to motivate the product rule of differentiation. Finally, a property of even and odd functions and their derivatives contradicts the misconception.

Mathematical Foci

Mathematical Focus 1

The slope of a secant line drawn through two close points on the graph of a function can approximate the derivative of the function between those points.

The derivative of any function represents the slope of the tangent to the graph of the function at every value of *x*. The slope of the tangent can be approximated by the slope of a secant line drawn through two points close to a particular value of x.

If $y' = 2x \cos x$ is the derivative of $y = x^2 \sin x$, then each value of $y' = 2x \cos x$ will represent the slope of the tangent to $y = x^2 \sin x$ at the corresponding value of *x* and can be approximated by a secant line. There follows a graph of the function and of a secant drawn through two points near x = 2.

f(x) = x²·sin(x)

Slope $\overleftrightarrow{AB}$ = 1.97

A: (1.93, 3.49)

B: (2.07, 3.76)
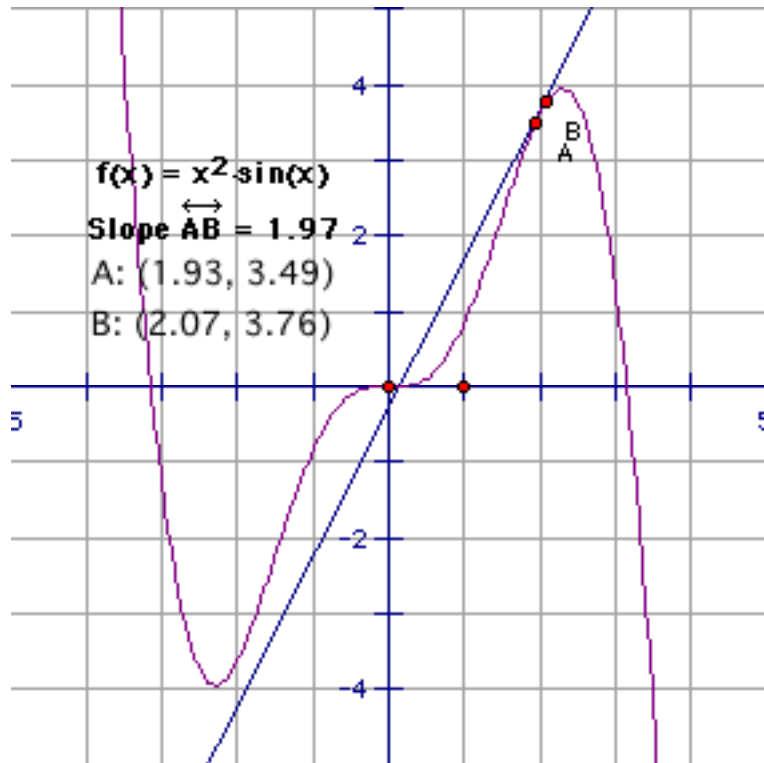
Consider the *x*-coordinates of points A and B, 1.93 and 2.07. Since they are close to the value of 2, the slope of the secant line AB should give a good approximation for the derivative of $y = x^2 \sin x$ at $x = 2$. The slope of this secant line is 1.97 and therefore, if $y' = 2x\cos x$ is in fact the derivative of $y = x^2 \sin x$, then the value of the derivative at $x = 2$ should be approximately 1.97. However, by substituting $x = 2$, we find $y'(2) = 2 \cdot 2\cos(2) \approx -1.665$. The values do not even agree in sign. Upon further inspection, one notes that the function is increasing on the interval around 2 and a negative derivative at the point x = 2 makes no sense.

Mathematical Focus 2

The product rule for differentiation necessarily follows from the definition of the derivative of a function.

The derivative of a function is defined as a limit of the slopes of secant lines drawn through two infinitely close points on the graph of the function. Thus, the derivative of a function $f(x)$ is given by $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. By substituting the function $x^2 \sin x$ for $f(x)$, we write:

$$f'(x) = \lim\limits_{h \to 0} \frac{(x+h)^2 \sin(x+h) - x^2 \sin(x)}{h}.$$

Using trigonometric identities and algebra, we arrive at the equivalent equation:

$$f'(x) = \lim_{h \to 0}\left[x^2 \cos(x) \cdot \frac{\sin(h)}{h} + 2x\sin(x) + 2x\cos(x)\sin(h) + h\sin(x) + h\cos(x)\sin(h)\right].$$

It follows, letting $h \to 0$ and recalling that $\lim_{h \to 0}\dfrac{\sin(h)}{h} = 1$, that:

$$f'(x) = x^2 \cos(x) \cdot 1 + 2x\sin(x) + 0 + 0 + 0.$$

Therefore,

$$f'(x) = x^2 \cos(x) + 2x\sin(x).$$

To demonstrate that $2x\cos x \neq x^2 \cos(x) + 2x\sin(x)$, we can evaluate both expressions for a particular value of x, say $x = \pi$ and get different results. When $x=\pi$, we have $2\pi \cos \pi \neq \pi^2 \cos \pi + 2\pi \sin \pi$, since the left side simplifies to 0 and the right side simplifies to 2π. So, $2x\cos x \neq x^2 \cos(x) + 2x\sin(x)$ and $y' = 2x\cos x$ cannot possibly be the derivative of $y = x^2 \sin x$.

In general, the definition of the derivative of a function implies the product rule. The function $k(x) = x^2 \sin x$ is a product of two functions, $f(x) = x^2$ and $g(x) = \sin x$. Therefore, $k(x) = x^2 \sin x$ can be written as $k(x) = f(x) \cdot g(x)$. Using the definition of the derivative, it follows that:

$$k'(x) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}.$$

After algebraic manipulation, we have,

$$k'(x) = \lim_{h \to 0}\left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}\right],$$

and by using the definition of the derivative, the limit expression simplifies to:

$$k'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

This result is perfectly general and is known as the product rule of differentiation. We can apply the product rule to the problem in the prompt, since the function $y = x^2 \sin x$ is a product of two functions $f(x) = x^2$ and $g(x) = \sin x$, where $f'(x) = 2x$ and $g'(x) = \cos x$. The derivative of $y = x^2 \sin x$ given by the product rule is $y = x^2 \cdot \cos x + (\sin x) \cdot 2x$, the same as was discovered directly from the definition of the derivative.

Mathematical Focus 3
*A function and its derivative cannot both be even, nor can both be odd.*

Functions that satisfy f(-x) = -f(x) are called *odd* functions and functions satisfying f(-x) = f(x) are called *even* functions. Odd functions have graphs that are symmetric around the origin while even functions have graphs symmetric around the y-axis. In the graphs below, one can see that the functions f(x) = sin x and f(x) = x are odd while f(x) = cos x and f(x) = x² are even.
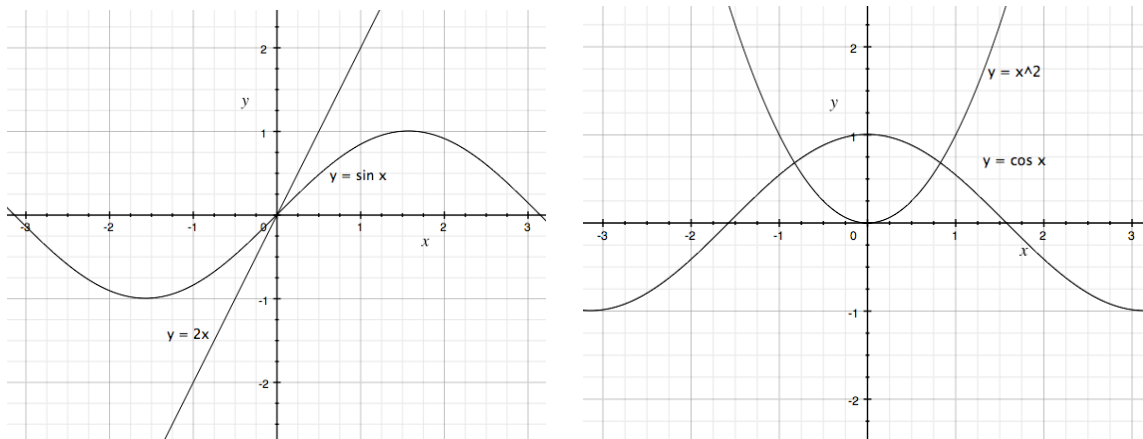
If a polynomial function is even, then the function takes on positive values (or negative values if the first term is negative) both for values of x approaching negative infinity and for values of x approaching positive infinity. Zooming out, the graph of the function looks like a mountain or a valley. The derivative of the function is the slope of its graph, and so it must be negative when x is very small and positive when x is large (or vice-versa if the first term of the polynomial is negative). This means that the derivative of an even polynomial function cannot be even. A similar argument shows that the derivative of an odd polynomial function cannot be odd.

The trigonometric functions do not exhibit such straightforward end-behavior, but the same relationship between a function and its derivative holds true: both cannot be even nor can both be odd. Consider the graph of the even function f(x) = cos x on the interval (-π, π). It has positive slope on (-π, 0) but negative slope on (0, π); therefore, the derivative of f(x) cannot be even. It's easy to see that the same argument works for any odd or even trigonometric function.

This property of odd and even functions and their derivatives can be applied to show that the derivative of $y = x^2 \sin x$ is not $y' = 2x \cos x$. By inspecting the graph of both functions, we see that both are odd. Thus one cannot be the derivative of the other.
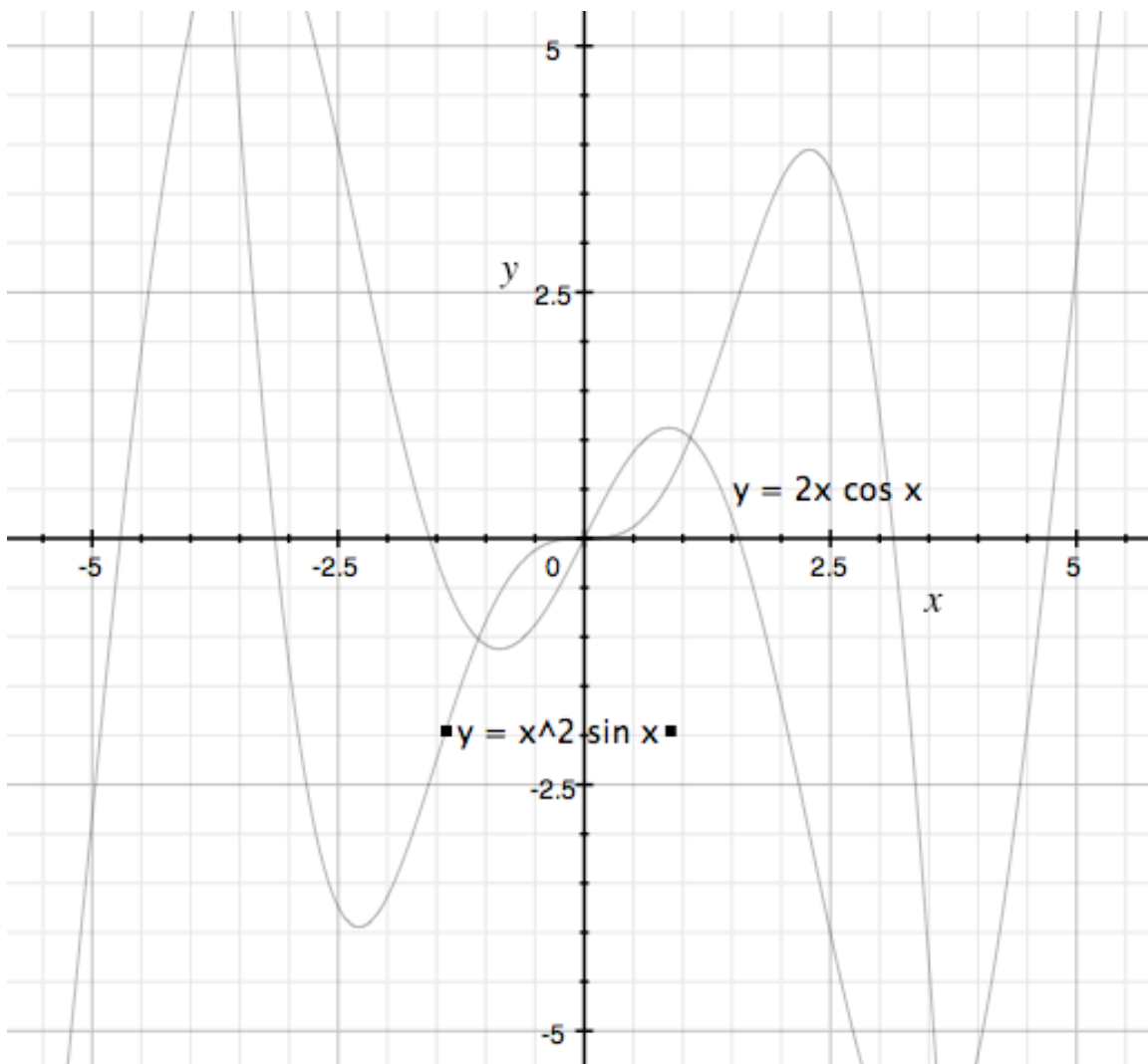
Indeed, an even easier argument does not rely on graphing or inspection. Since we know that f(x) = sin x and f(x) = 2x are odd functions and that f(x) = cos x and f(x) = x² are even functions, and because the product of an even and odd function is always an odd function, it must be that $f(x) = x^2 \sin x$ and $f(x) = 2x \cos x$ are both odd functions. Thus one cannot be the derivative of the other.

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