MAC-CPTM Situations Project

Situation 49: Similarity

Prepared at Penn State University
Mid-Atlantic Center for Mathematics Teaching and Learning
15 July 2005 – Evan McClintock, Sue Peters, Donna Kinol

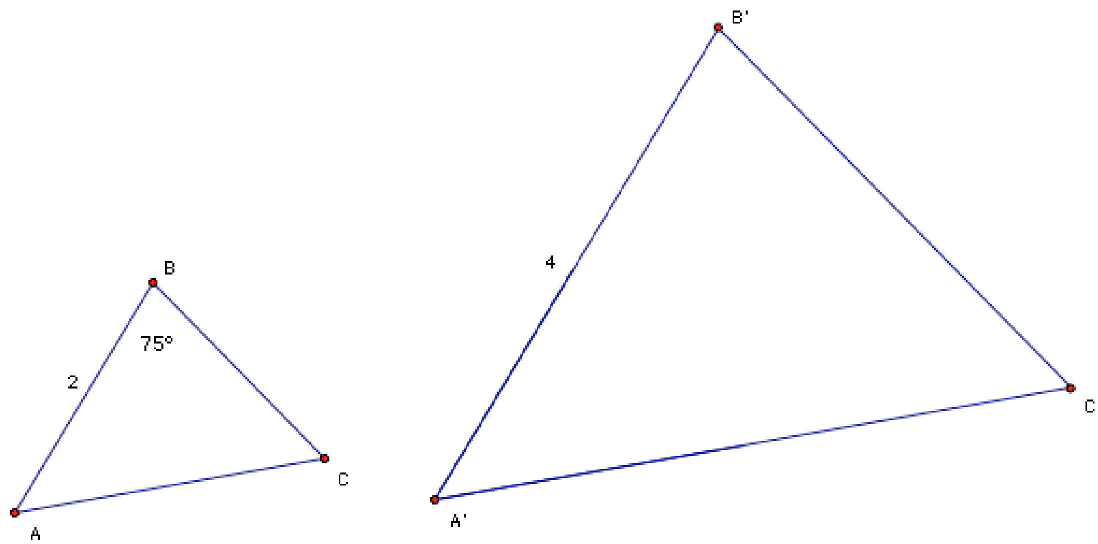
Edited at Penn State University
Mid-Atlantic Center for Mathematics Teaching and Learning
10 May 2006 – Evan McClintock, Shari Reed, Heather Godine

Edited at University of Georgia
Center for Proficiency in Teaching Mathematics
10 February 2007 – Erik Tillema

Edited at Penn State University
Mid-Atlantic Center for Mathematics Teaching and Learning
14 February 2007 – Heather Godine
23 February 2007 – Rose Mary Zbiek, M. Kathleen Heid

Prompt

In a geometry class, students were given the following diagram depicting two acute triangles, ΔABC and $\Delta A'B'C'$, and students were told that $\Delta ABC \sim \Delta A'B'C'$. From this, a student concluded that $m\angle B' = 150°$.



Commentary

Under a geometric similarity transformation, angle measure is preserved and the ratio of the measures of corresponding distances is constant. The first two foci pertain to properties of polygons preserved under similarity. The third and fourth foci appeal to central dilation, a specific similarity transformation. The fifth focus utilizes geometric construction. The foci incorporate a variety of approaches: geometric, graphical, and symbolic.

Mathematical Foci

Mathematical Focus 1

Length measures of corresponding parts of similar figures are related proportionally with a constant of proportionality that may be other than 1, but angle measures of corresponding angles of similar figures are equal.

Similar triangles have corresponding angles that are congruent and corresponding sides that are in proportion. The diagram in the prompt depicts two similar triangles, ΔABC and $\Delta A'B'C'$, with AB = 2 and A′B′ = 4. A ratio of the lengths of the sides AB and A′B′ can be used to determine the relative lengths of the sides of $\Delta A'B'C'$ as scaled sides of ΔABC or vice versa. In particular, since A′B′ = 2AB and ΔABC and $\Delta A'B'C'$ are similar, it must be true that A′C′ = 2AC, and B′C′ = 2BC. While the constant of proportionality, 2, can be used to find relative lengths of corresponding sides of these similar triangles, it does not apply to the measures of the angles. In particular, doubling the measure of each of the angles of ΔABC would result in the sum of the measures of the angles of $\Delta A'B'C'$ being 360°, which is not possible because the sum of the measures of the angles of any triangle is 180°.

Mathematical Focus 2

When comparing corresponding parts of figures and their images under similarity transformations, angle measure is preserved and the length of an image is the product of the ratio of similitude and the length of the original figure. With ratio of similitude 1, isometries are a subset of similarity transformations.

Transformations in which shape is preserved but size is not necessarily preserved are similarity transformations. Given ΔABC, consider $\Delta A'B'C'$ to be the image of a similarity transformation of ΔABC. A geometric similarity transformation is an angle-preserving function such that all distances are scaled by a constant ratio, $k \neq 0$. For the given similarity transformation, the lengths of sides of $\Delta A'B'C'$ are double the length of the corresponding sides of ΔABC, and the angles of $\Delta A'B'C'$ are congruent to the corresponding angles of ΔABC.

Dilations whose center is the origin in a coordinate plane are similarity transformations of the form $F((x, y)) = (kx, ky)$ for some constant ratio $k \neq 0$. In general, if $|k| < 1$, the mapping results in a contraction, for which the resulting image is smaller than its pre-image. If $|k| > 1$, the mapping results in an expansion, for which the resulting image is larger than its pre-image. If $k = 1$, the mapping is the identity transformation under composition of transformations, for which the resulting image is the same size and shape as its pre-image.

Transformations in which shape and size are preserved are known as isometries, for which the resulting image is congruent to the pre-image. There are five distinct types of isometries: identity, reflection, non-identity rotation, non-identity translation, and glide reflection. In the coordinate plane, any figure may be mapped to a similar figure by a composition of dilations and isometries. The constant of proportionality is the product of the ratios of similarity. When one figure is mapped to its image by only isometries, the product of the ratios is a power of 1 and the figures are congruent.

Mathematical Focus 3

For a triangle inscribed in a circle and its dilation through the center of the circle, the relationship between the inscribed angle and the length of the intercepted arc can be used to show that angle measure is preserved under dilation.

Using GSP, a dynamic diagram can be created to illustrate that shape and angle measure are preserved for similar triangles for which one can be represented as the expansion or contraction of the other triangle (with the center of the circle that inscribes that triangle as the center of dilation). Consider a triangle inscribed in a circle centered at the origin. Using polar coordinates, the coordinates of any point on the circle are (r,θ), where r is the radius of the circle and θ is the measure of the angle in standard position formed by the x-axis and a ray from the origin to a point on the circle. For any angle in the triangle, the measure of the angle is equal to half the length of the intercepted arc divided by the radius. Thus, angle measure is a function of arc length and radius, namely, $m(a,r) = a/(2r)$. Since the length of an arc is equal to the product of the radian measure of the arc angle and the length of radius of the circle, the ratio of half the length of the arc to the radius of the circle will be equal to the radian measure of the angle in the triangle intercepting that arc, regardless of the length of the radius. Therefore, as a circle centered at the origin is expanded or contracted, the measure of any angle in the inscribed triangle remains constant. Thus, angle measure is preserved.

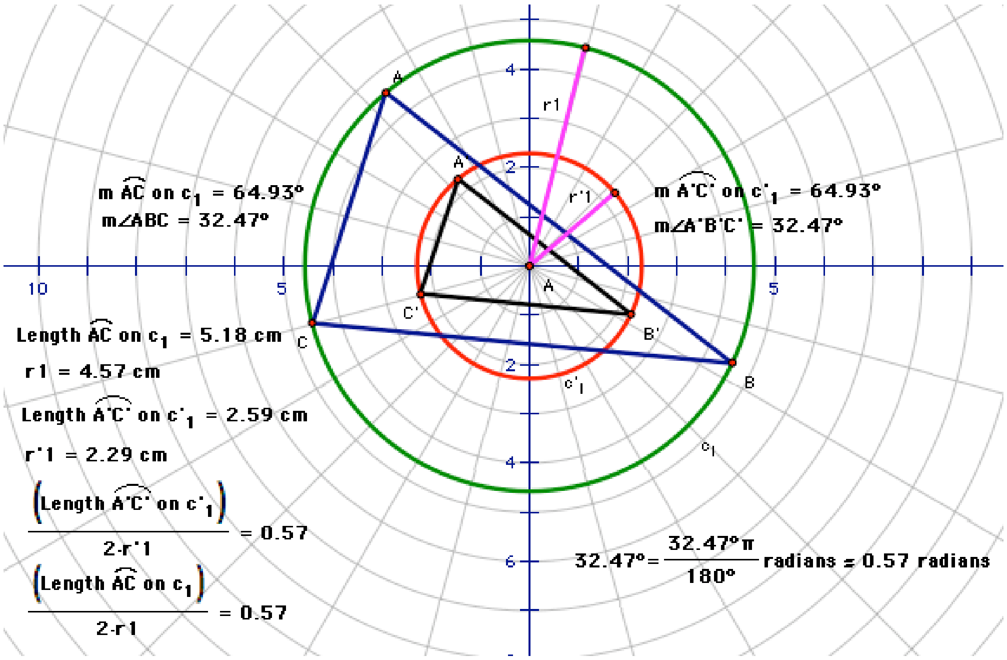


Figure 2

Mathematical Focus 4

Similar triangles have corresponding angles that are congruent and have corresponding sides that are proportional in length.

Consider two triangles, ΔABC and ΔDEF such that $\angle A \cong \angle D, \angle B \cong \angle E, and \angle C \cong \angle F$ (Figure 3).

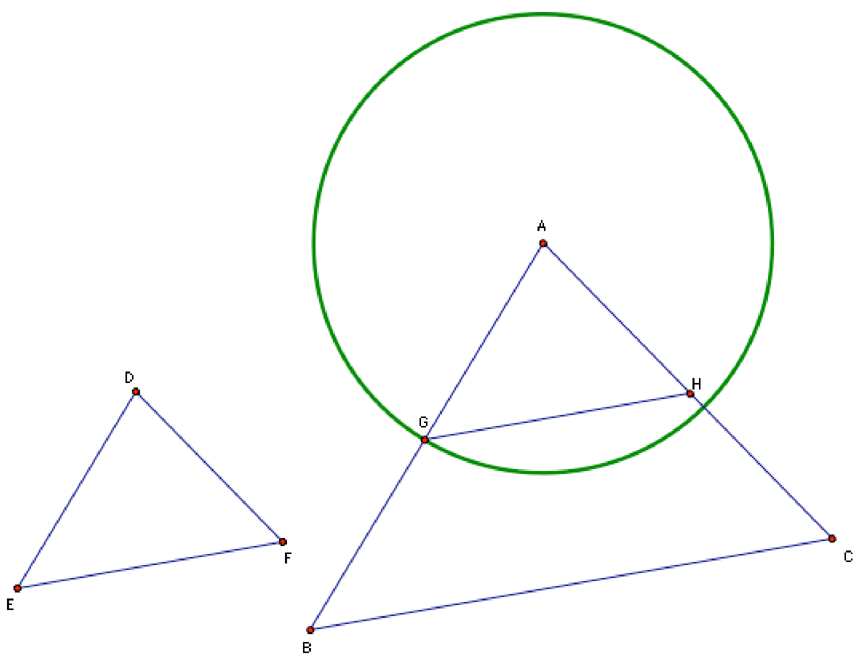


Figure 3

Construct a circle centered at A with radius $\overline{DE}$. Let G be the point of intersection of the circle and $\overline{AB}$. Construct a line through G parallel to $\overline{BC}$. Let H be the point of intersection of the parallel line and $\overline{AC}$. Since $\overleftrightarrow{GH} \parallel \overleftrightarrow{BC}$, corresponding angles are congruent, so ∠AGH ≅ ∠E. Therefore, ΔAGH ≅ ΔDEF by the ASA theorem of congruence. Since $\overleftrightarrow{GH} \parallel \overleftrightarrow{BC}$, and parallel lines divide transversals proportionally, $\frac{GB}{AG} = \frac{HC}{AH} \Rightarrow \frac{AB}{AG} = \frac{AC}{AH} \Rightarrow \frac{AB}{AC} = \frac{AG}{AH}$, which is equivalent to $\frac{AB}{AG} = \frac{AC}{AH}$. Since AG = DE and AH = DF, by substitution, $\frac{AB}{DE} = \frac{AC}{DF}$. Using similar reasoning, it can be shown that $\frac{BC}{EF} = \frac{AB}{DE}$. Therefore, by the transitive property of equality: $\frac{BC}{EF} = \frac{AC}{AH}$. Thus, it can be concluded that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

Post Commentary

In all of the foci, we find some type of ratio. It might be a ratio of similarity, a ratio among side lengths, or a ratio of proportionality. This common ratio lies at the heart of why size but not shape changes. When the ratio corresponds to 1, congruence emerges as special case of similarity.