MAC-CPTM Situations Project

Situation 50: Connecting Factoring with the Quadratic Formula

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Prompt

Mr. Jones just completed a unit on factoring quadratic polynomials and is now beginning a unit on the quadratic formula. One student asks if there is a direct connection between the two concepts. Mr. Jones wonders about the different ways he can answer the question.

Commentary

The topic of solving quadratic equations covers three processes: 1) factoring, 2) completing the square; and 3) using the quadratic formula. Although each process can be used to solve any quadratic equation, sometimes one is more efficient than the others. In the discussion that follows, the benefits of each process will be outlined as well as the connections among them. In addition, a geometric model will be offered to support each algebraic explanation.

Since any quadratic polynomial can be represented as a product of two binomials, the polynomial can be modeled geometrically as the area of a rectangle with sides the length of the binomials. Quadratic polynomials of the form $x^2 + bx + c$ can be represented as a product of the form $(x+m)(x+n)$, where m,n may have integral and real values. Partitioned area models of rectangles emphasize the relationship between the sum $x^2 + bx + c$ and the product $(x+m)(x+n)$.

Mathematical Foci

Mathematical Focus 1

If the quadratic expression factors over the integers, then the factored expression can easily and directly be modeled as a rectangle with sides (x+m) and (x+n), i.e. $x^2 + bx + c = (x+m)(x+n)$ *for some* $m,n \in \mathbb{Z}$.

The most obvious connection between factoring and the quadratic formula is the zero product property which is discussed in depth in Situation 45. In essence, the solutions to the quadratic formula are the roots used in the factors set up by the zero product property. For example, in $x^2 - 12x + 20 = 0$, the quadratic formula will yield the solutions $x = -10$ *and* $x = -2$. These in turn become $(x+10)(x+2) = 0$.

Consider the quadratic polynomial $x^2 + 3x + 2$, and a partitioned area model (figure 1).

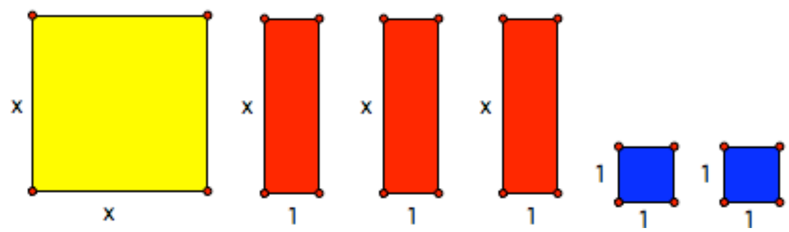

Figure 1

The process of factoring involves re-expressing a sum as a product. Hence, by composing a rectangle from the partitioned area model, the sum of the areas can be expressed as a product of the lengths of the sides of the rectangle (figure 2).

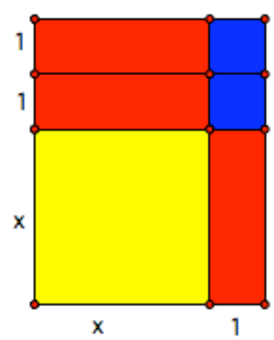

Figure 2

Since the area of the rectangle can be represented as the product of the lengths of the sides of the rectangle or as the sum of the partitioned areas, we can state

$$x^2 + 3x + 2 = (x+1)(x+2).$$

Factoring is more efficient when the factors are integers, but completing the square also works in these cases.

$$x^2 + 3x + 2 = 0$$

Step 1) $$x^2 + 3x + \left(\frac{3}{2}\right)^2 + \left[2 - \left(\frac{3}{2}\right)^2\right] = 0$$

Step 2) $$(x + \frac{3}{2})^2 + \frac{1}{4} = 0$$

Step 3) $$x = -\frac{3}{2} \pm \frac{1}{2} = -1 \text{ or } -2$$

Since x = -1 and x = -2, the Root-Factor Theorem of Algebra tells us that (x+1) and (x+2) are factors of the polynomial expression $x^2 + 3x + 2$. The result is the same with either method. Indeed, completing the square relies on the fact that perfect square trinomials can be factored easily.

The results are the same with either method.

Mathematical Focus 2

If the quadratic expression has real-valued factors that are not integers, then the method of completing the square is an efficient method to find them, and partitioned area can geometrically model the expression and its factors.

Completing the square finds a perfect square trinomial less a squared constant that is equal to the original expression. The factors of the original expression are the square root of the trinomial plus and minus the constant. Geometrically, these factors correspond to the sides or a rectangle formed by rearranging the two parts of a square with a smaller square removed. Consider the quadratic polynomial $x^2 - 9$ and a partitioned area model (figure 3).

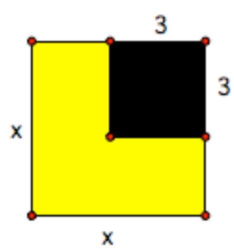

Figure 3

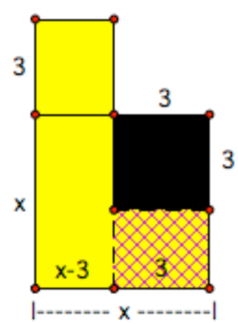

Figure 4

By cutting, rearranging, and pasting back together the area in Figure 3, a rectangle of the same area can be made, Figure 4.

Since the area of the rectangle can be represented as the product of the lengths of the sides of the rectangle or as the sum of the partitioned areas, $x^2 - 9 = (x+3)(x-3)$. This rearrangement of the area of a square less a square into a rectangle is at the heart of the method of completing the square.

Consider the following expression that cannot be factored over the integers, the quadratic polynomial $x^2 + 6x + 7$. We can find the roots by completing the square.

$$x^2 + 6x + 7 = 0$$

Step 1) $x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 7 = 0$

Step 2) $(x+3)^2 - 2 = 0$

Step 3) $x = -3 \pm \sqrt{2}$

Using the Root-Factor Theorem once more, we conclude that

$$x^2 + 6x + 7 = \left(x + 3 + \sqrt{2}\right)\left(x + 3 - \sqrt{2}\right).$$

It is important to note that the process of completing the square is built upon factoring – that the "square" being completed is a perfect square trinomial which is later written in factored form. In the above example, the perfect square trinomial is introduced in step 1 and factored in step 2.

The method of completing the square can be modeled geometrically. The original quadratic polynomial $x^2 + 6x + 7$ is represented by the partitioned area model in Figure 5.

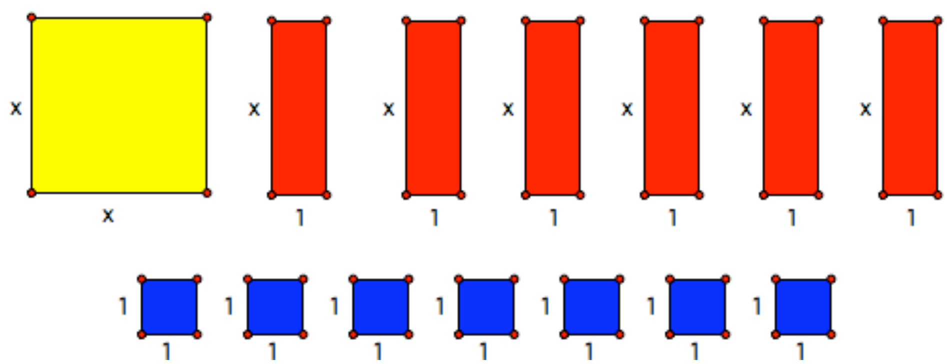

Figure 5

The partitioned area model is re-expressed in Figure 6; however, a rectangle has not been composed.

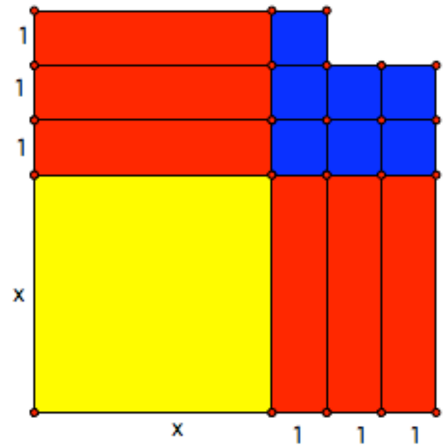

Figure 6

Since $x^2+6x+7=\left(x^2+6x+9\right)-2=\left(x+3\right)^2-2$, the area in Figure 9 can be cut a part and pasted back together into the area in Figure 7. Now, the area of the original quadratic expression is represented as a square representing $\left(x+3\right)^2$ with a small square representing two units of area removed.

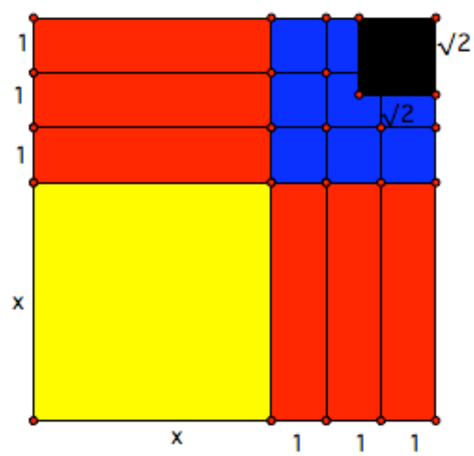

Figure 7

To factor the quadratic polynomial x^2+6x+7, compose a rectangle from the partitioned area model (Figure 8)

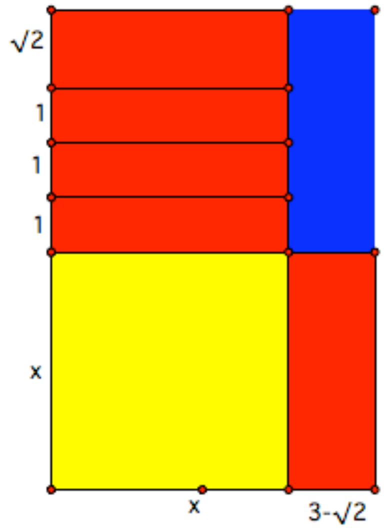

Figure 8

Since the area of the rectangle can be represented as the product of the lengths of the sides of the rectangle or as the sum of the partitioned areas,

$$x^2 + 6x + 7 = \left(x + 3 + \sqrt{2}\right)\left(x + 3 - \sqrt{2}\right).$$

This is the same result that was achieved (and by the same process) when completing the square.

If the quadratic equation has complex roots, then, although the area partition model no longer makes sense, the method of completing the square still gives correct results.

Consider the quadratic polynomial $x^2 + 4x + 5$,

$$x^2 + 4x + 5 = 0$$

Step 1) $\quad x^2 + 4x + (2)^2 + \left[5 - (2)^2\right] = 0,$

Step 2) $\quad (x + 2)^2 + 1 = 0,$

Step 3) $\quad x = -2 \pm i.$

Using the quadratic formula would give the same results, a fact explained in Focus 3.

Mathematical Focus 3

The process of completing the square is used to derive the quadratic formula. In addition, the solutions given by the quadratic formula can be represented by a partitioned area model.

Consider the quadratic equation written in general form...

$$ax^2 + bx + c = 0.$$

The derivation of the quadratic formula from this general equation involves the use of completing the square:

$$1)\ x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

$$2)\ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2,$$

$$3)\ \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2,$$

$$4)\ \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2},$$

$$5)\ x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}},$$

$$6)\ x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

$$7)\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note that the process of completing the square, as demonstrated in steps two and three, is built upon the idea of factoring. In turn, it can be argued that the quadratic formula is built upon factoring as well.

The quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is used to determine solutions to a quadratic equation of the form $ax^2 + bx + c = 0$. When $a = 1$, the quadratic equation becomes

$x^2 + bx + c = 0$ and applying the quadratic formula gives solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2},$$
$$x = \frac{-b}{2} \pm \sqrt{\frac{b^2 - 4c}{4}},$$
$$x = \frac{-b}{2} \pm \sqrt{\frac{b^2}{4} - c}.$$

The solutions can be used to write the equation in factored form based upon the Zero Product Property:

$$x^2 + bx + c = 0,$$

$$(x + \tfrac{b}{2} + \sqrt{\tfrac{b^2}{4} - c})(x + \tfrac{b}{2} - \sqrt{\tfrac{b^2}{4} - c}) = 0.$$

Consider the quadratic polynomial $x^2 + bx + c$ and a partitioned area model (Figure 9).

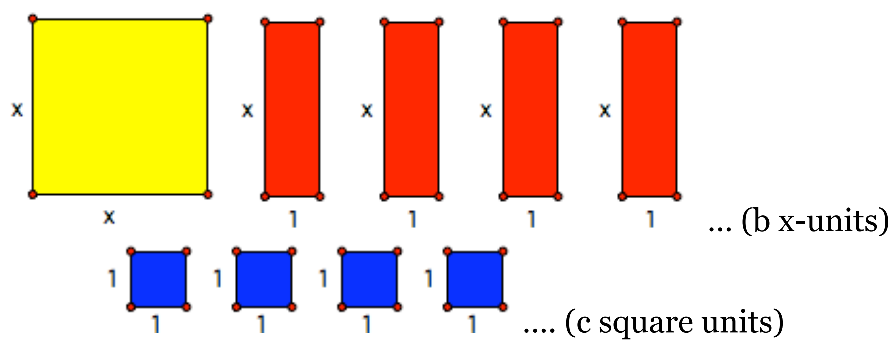

Figure 9

To factor the quadratic polynomial $x^2 + bx + c$, compose a rectangle from the partitioned area model (Figures 10 and 11).

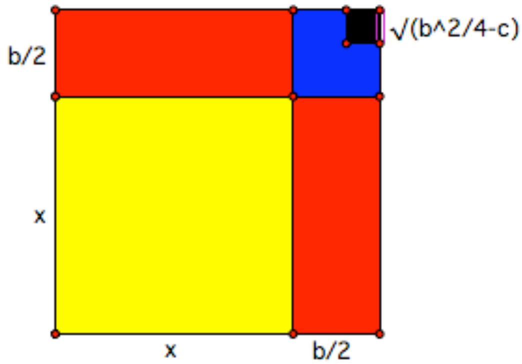

Figure 10

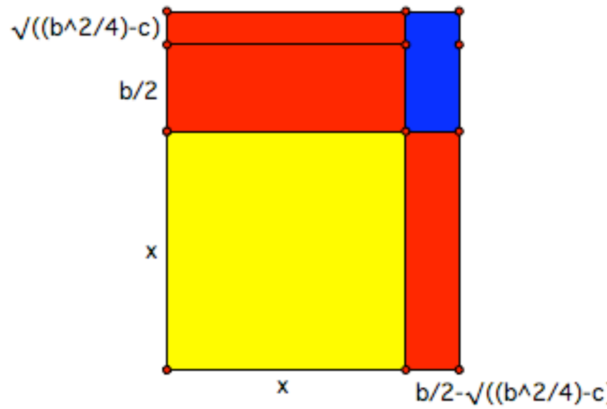

Figure 11

Since the area of the rectangle can be represented as the product of the lengths of the sides of the rectangle or as the sum of the partitioned areas, then we can state that $x^2 + bx + c = (x + \frac{b}{2} + \sqrt{\frac{b^2}{4} - c})(x + \frac{b}{2} - \sqrt{\frac{b^2}{4} - c})$.