

Situation 51: Proof by Induction
Prepared at the University of Georgia
Center for Proficiency in Teaching Mathematics
10/13/06-Erik Tillema

Prompt

A teacher of a calculus course gave her students the opportunity to earn points by proving various algebraic formulas by induction. Only one student in the class was able to prove any of the formulas. After his third proof to the class on three consecutive days, another student complained: “I don’t get what he is proving and besides that how do you get the algebraic formulas to start with”. The teacher realized that she needed to do some background work on proofs by induction as well as give some motivation for where the formulas come from in order to make the playing field a bit more level for her students.

Commentary

The foci look at several different aspects that might be important to developing proofs by induction as well as ways of generating conjectures that can be proved using proof by induction. Specifically, focus 1 looks at a common image for what proof by induction accomplishes. Focus 2 examines a problem context in which a recursion relationship might be formed and used to formulate an algebraic statement and a subsequent proof by induction. Focus 3 gives several examples of how figurate representation of numbers can lead to conjectures about algebraic formulae and subsequent proofs by induction. Focus 4 suggests examples where one of the two conditions given in focus 1 fail. It is intended as an analysis of the structure of proofs by induction and is more in the province of metamathematics than mathematics. Further, it suggests why starting with algebraic formulas (true or false) as opposed to problem situations may not be a good starting place for proofs by induction because in doing so one starts with an analysis of a type of proof rather than the generation of a proof itself.

Focus 1

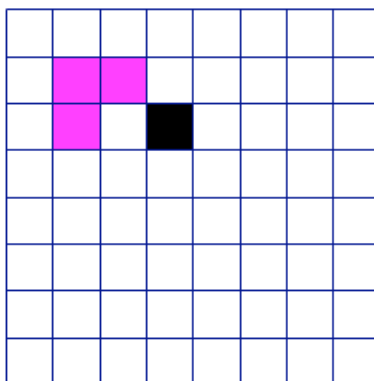
It may be useful to provide an image of what exactly induction does. One good image might be of a ladder that is infinitely tall (used to represent the integers). In doing a proof by induction, we first have to check if we can get on the first rung of the ladder (which corresponds to checking if the base case is true). Once we know we can get on the ladder we have to check that if we are on any step of the ladder we can proceed to the next step (which corresponds to showing that if we are on the k^{th} rung that we can move to the $(k+1)^{\text{st}}$ rung or that if the proposition holds for k it will hold for $k+1$). By proving these two things, we are now free to assume that we can get to any rung of the ladder (or that the proposition is true for all k). The reason this assumption holds true is we can get on the first rung of the ladder (from the first part of our induction proof) and then we will be able to get on the second rung (by the second part of our induction proof) but

then we know that we can get on the second rung and therefore we can get on the third rung (by the second part of our induction proof), etc.

Focus 2¹

Many recursion formulas create nice problem solving opportunities that might lead to algebraic formulations of a situation and subsequent proofs by induction. One such problem situation is shown below.

Suppose you have a $2^n \times 2^n$ chessboard. Demonstrate that if you remove one square (shown in black) that you can cover the board with triominoes (shown in purple).

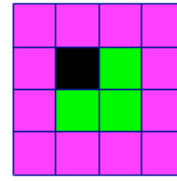
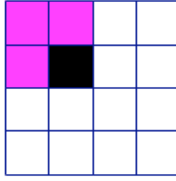


In the case shown above, there is an 8×8 chessboard. It is easy to show that it is possible to find a covering of the chessboard but it is a little bit difficult to see a pattern in a particular covering of a chessboard of this size. This difficulty provides a good opportunity to use Polya's (1945) strategy of looking at a simpler instance of the problem. To this end, it might be useful to examine a 2×2 and 4×4 chessboard. In the 2×2 case it is pretty obvious that we will be able to cover the chessboard without any difficulty. That is, we can place a triomino as we please and there will be exactly one space left over.



We might next look at a 4×4 board and notice that it is composed of exactly four 2×2 chessboards (and further that there are four 4×4 chessboards in an 8×8 chessboard). In making this observation, we now can establish a recursion relationship between each successive chessboard. Also, we might observe that our 4×4 board will have exactly one 2×2 chessboard with the space taken out of it and three 2×2 chessboards with exactly one space not covered which we can always cover with exactly one triomino (shown on the left).

¹ This problem was a part of Dr. Patricia Wilson's EMAT 6650 course.



Having made these observations, we can set up a recursion relationship between successive boards and make a more formal proof by induction. Note in this case we might simply make a proof by induction by writing out our observations.

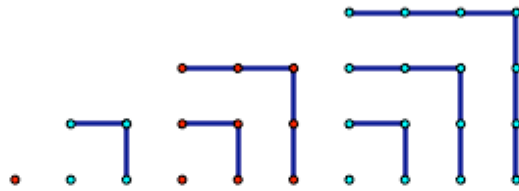
On the other hand, we might use this problem to develop the formulation of an algebraic statement like $3 \mid 2^{2^n} - 1$ for all n . In order to think of this algebraic formulation, we have to notice that the chessboard contains 2^{2^n} squares and that removing one of the squares gives us $2^{2^n} - 1$ squares to be covered. Next we need to observe that the triomino contains 3 squares. Further, we have to see that the spatial arrangement of the triomino and the board do not have any effect on the formulation of the problem. This insight may perhaps be the most difficult thing to see since it appears that our initial argument depended on the spatial configuration of both the board and the triomino. To overcome this difficulty, we might convince ourselves that the spatial configuration does not matter in the formulation by imagining cutting the board at each row and putting the rows end to end. Next, we might imagine straightening out the triomino. I have shown this below for the 4 x 4 chess board.



We can once again make a recursive argument to show that we can cover all but one of the squares of our transformed board with our transformed triomino. From this spatial configuration, we may become more convinced that we are asking ourselves a question of division. That is, how many threes will “go into” a row of length $2^{2^n} - 1$.

Focus 3

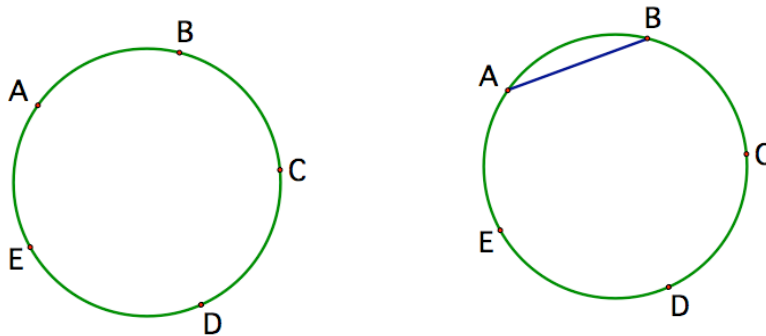
Configurations of figurate numbers are often used to find identities that then can be proved by induction. We might first consider making a figurate representation for the square numbers. In looking at such a configuration, we might see the pattern shown below:



The configuration seems to suggest that $n^2 = 1 + 3 + \dots + (2n-1)$. Having found a conjecture about the formula, we can then use our conjecture to try a proof by induction.²

A second situation that uses figurate numbers and extends Situation 38, which demonstrates several ways to get an intuitive feel for why the sum of the first n integers is equal to $\frac{n(n+1)}{2}$, is

finding the sum of the first n sums of integers. Algebraically, that would be if $T_1 = 1, T_2 = 1 + 2, T_n = 1 + 2 + \dots + n$, then we want to find $T_1 + T_2 + \dots + T_n$. We can introduce such a problem by finding both an additive and multiplicative solution to the following problem. Suppose that there are five points on a circle. How many possible triangles could you make?



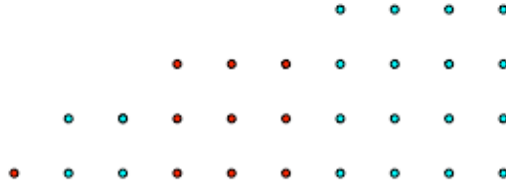
We might structure the problem additively by thinking about finding the number of triangles we can make with AB as the base of our triangle (3), BC as the base of the triangle (2), and CD as the base of a triangle (1). To think about it multiplicatively, we can think about choosing sets of three points from five or $\binom{5}{3}$. After solving this for a few cases, we might conjecture that

$T_1 + T_2 + \dots + T_n = \binom{n+2}{3} = \frac{1}{6}(n+2)(n+1)(n)$ and use this problem context to make a proof by induction of this identity.³

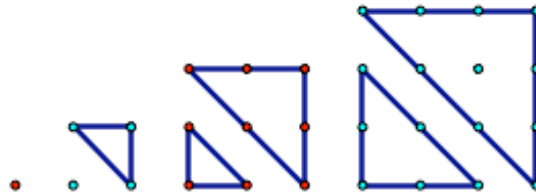
We might then consider using this identity to begin looking for an identity for $1^2 + 2^2 + \dots + n^2$. We again make a figurate representation of the problem to help us see what the identity might be by making the first four square numbers.

² It is worth looking for a way to represent the sum of the first n even numbers as well.

³ An extension of this problem is to find an additive and multiplicative way of thinking about the number of diagonals that can be drawn given a convex n -gon.

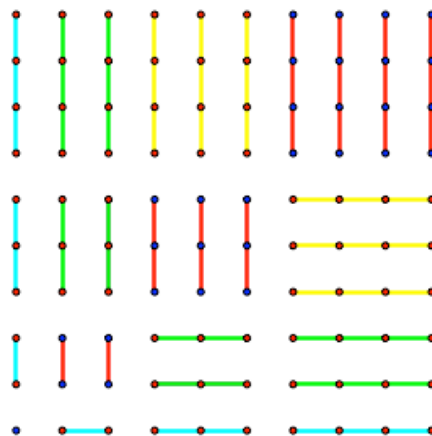


Next knowing that we have a formula for the sum of the first n triangular numbers, we might see if we can break these into triangular numbers as shown below.



From our example of the sum of the first four square numbers, we might observe that the sum of the first n square numbers is $2T_1 + 2T_2 + \dots + 2T_{n-1} + T_n = \binom{n+2}{3} + \binom{n+1}{3}$. With a little algebraic manipulation, we get that $1^2 + 2^2 + \dots + n^2 = \frac{1}{6}(2n+1)(n+1)(n)$ and another formula that we conjecture to be true and might prove its validity using proof by induction.

A final problem might be to look at the sum of the first n integers squared or $(1 + 2 + \dots + n)^2$. We again might want to make a figurate representation of the case when $n = 4$ as shown below.



We might ask ourselves how many fours we have in our picture above. We see that horizontally there are $\binom{5}{2}$ fours (the sum of the first four integers worth) and vertically there are $\binom{4}{2}$ fours

(the sum of the first three integers worth), giving a total of $4\left(\binom{5}{2} + \binom{4}{2}\right)$ dots. With a little

algebraic simplification, we see that there are 4^3 dots in the outer most region of our configuration. A similar process shows us that in computing the dots in the region with the threes gives us 3^3 dots, etc. Giving us a conjecture for the following identity: $(1 + 2 + \dots + n)^2 = 1^3 + 2^3 + \dots + n^3$. We then can prove the formula we think to be true by induction.

Focus 4

We might want to examine a little bit further the structure suggested by the ladder image of focus 1 to demonstrate the importance of showing that both conditions hold. That is, we are able to both get on the ladder (show that the base case holds true) and that once we are on the ladder we are able to move from one rung to the next rung (show that assuming $n = k$ is true implies $n = k + 1$ is true). Many of the examples where one of these assumptions fails is somewhat trivial but nonetheless they can give us some feeling for the importance of both assumptions.

For instance, if we consider the following inequality $5(2^n) > 6^n$ we see that for $n = 1$ the inequality holds true (we can get on the ladder). So we might assume that it is true for $n = k$ and try to show that this assumption implies that it is true for $n = k + 1$. So we want to show that $5(2^{k+1}) > 6^{k+1}$ given that we assume $5(2^k) > 6^k$. We can use the inductive hypothesis to form the following inequality $5(2^{k+1}) = 2(5(2^k)) > 2(6^k)$ and we would like to be able to conclude that $2(6^k) > 6(6^k) = 6^{k+1}$ to complete our proof. However, this argument is false and seeing no way to proceed we might return to our assumption to check it. So $5(2^n) > 6^n$ if we let $n = 2$ we immediately see that our assumption is false. In this case, we were able to get on the first step of the ladder but not able to move from rung to rung.

We might now consider the inequality that $3^n > 5(3^n)$. Here we might begin by assuming that it is true for $n = k$ and try to prove that it is true for $n = k + 1$. So in doing so we come up with the following, $3^{k+1} = 3(3^k) > 3(5(3^k)) = 5(3^{k+1})$, where the inequality holds true based on our inductive hypothesis (we are able to move from rung to rung). However, we see that in the case of $n = 1$ the inequality does not hold and so we are unable to actually get on the ladder although were we able to get on the ladder we could move from rung to rung. These examples illustrate that both conditions need to be met in order for a proof by induction to work. They also suggest that it is worth checking *several* cases when proving something by induction.

However, checking several cases is not a substitute for the proof as can be seen by the following inequality $3^n > n!$. Here we can check the first six cases and see that the inequality holds true but when $n = 7$ the inequality becomes false. We can convince ourselves that for any a the inequality $a^n > n!$ will eventually become false although we can make the n for which the inequality becomes false arbitrarily large so that we can find instances where the first one thousand cases will hold true, the first one million, etc. Such an example suggests that our proof by induction in

fact contains deductive reasoning. That is, we are not drawing a conclusion that is a generalization based solely on repeated observations (inductive reasoning) but rather we are establishing two premises and stating that if both premises hold true we can assert our conclusion (deductive reasoning). The three examples in this focus taken together suggest the relationship of the premises to the conclusion as well as the role that observation plays in relation to a proof by induction.