CAS Situation 2: Absolute Value and Square Roots
Prepared at Penn State
Mid-Atlantic Center for Mathematics Teaching and Learning
060919 – Heather Godine, Shiv Karunakaran, Evan McClintock

Prompt

A talented 7th-grade student was working on the task of producing a function that had certain given characteristics. One of those characteristics was that the function should be undefined for values less than 5. Another characteristic was that the range of the function should contain only non-negative values. In the process, he defined $f(x)=\left|\sqrt{x-5}\right|$ and then evaluated $f(-10)$. The result was 3.872983346. He looked at the calculator screen and whispered, "How can that be?"

Commentary

Mathematical Foci 1, 2, and 3 consider the student-generated function represented by $f(x)=\left|\sqrt{x-5}\right|$ from the perspective of substituting values for x to determine outputs. In contrast, Mathematical Focus 4 considers the student-generated function represented by $f(x)=\left|\sqrt{x-5}\right|$ from the perspective of a mapping of elements of a set A to elements of a set B such that each element in A maps to a single element in B.

Mathematical Focus 1 and Mathematical Focus 2 consider necessary mathematics that underlie the perspective of substituting values for x into a function to determine the outputs of the function. In particular, Mathematical Focus 1 describes how representing real numbers requires only a one-dimensional system, but representing complex numbers requires a two-dimensional system. Mathematical Focus 2 describes geometric interpretations of absolute value on the complex plane and the real line. Mathematical Focus 3 verifies that $f(-10)=3.873$ by considering the student-generated function $f(x)=\left|\sqrt{x-5}\right|$ as a composition of two functions, and in this case domain becomes a problematic issue. By considering function from the perspective of a mapping, what becomes central to Mathematical Focus 4 is the notion that any number of functions could represent a particular set of ordered pairs.

Mathematical Foci

Mathematical Focus 1

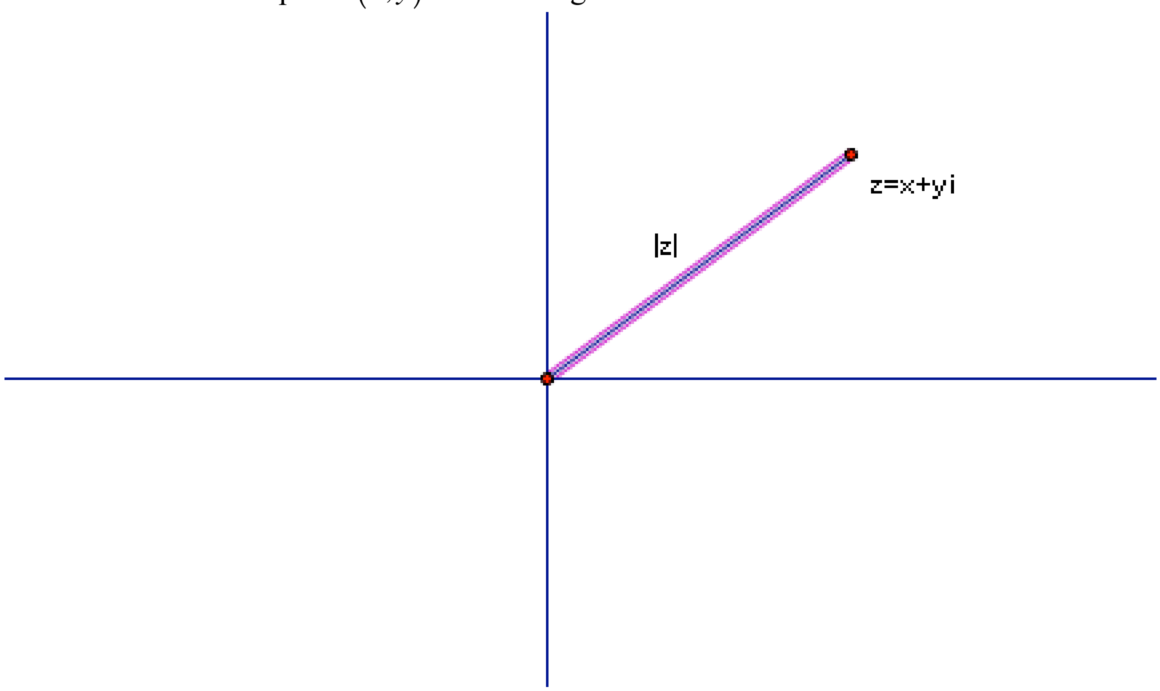
Representing real numbers requires only a one-dimensional system, as we can represent all of the real numbers on a single line. In this way, a real number, x, can be represented by a unique point on the real number line. On the other hand, representing complex numbers requires a plane, where the real numbers are contained on one line and the imaginary numbers are contained on another line perpendicular to the real line. In this way, a complex number $z=x+yi$ can be represented uniquely by a point having coordinates (x, y) on the complex plane. In contrast, the ordered pair (x, y) on the Cartesian coordinate plane $\Re \times \Re$ represents a coordination of the numbers $x, y \in \Re$, not a single real number.

By definition, complex numbers are of the form $z = x + yi$, where x and y are real numbers. Therefore, z is a real number if and only if $y = 0$. In this way, the real numbers are a complete proper subset of the complex numbers.

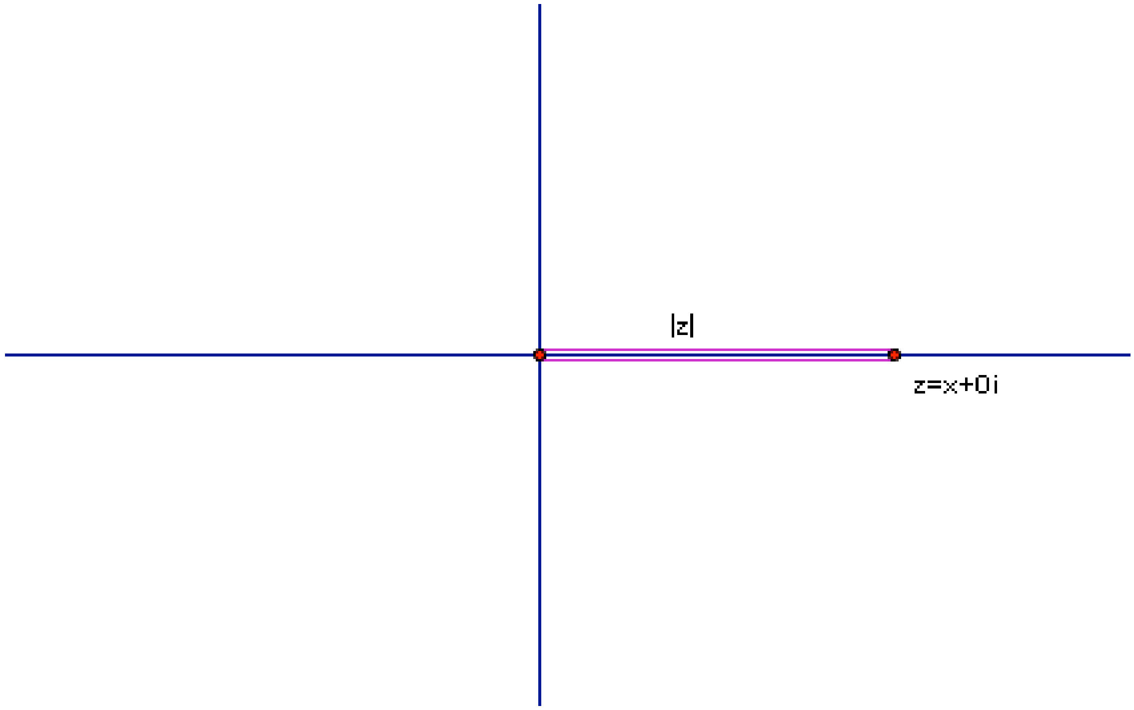
Mathematical Focus 2

The absolute value of a complex number

As observed before, complex numbers $z = x + yi$, where x and y are real numbers, can be represented by the point (x, y) on the complex plane. The absolute value of the complex number $z = x + yi$ is given by $|z| = \sqrt{x^2 + y^2}$. A geometric interpretation of $|z|$ is the distance between the point (x, y) and the origin.



When $y = 0$, z is a real number, and a geometric interpretation of $|z| = \sqrt{x^2 + 0^2}$ is the distance between the point $(x, 0)$ and the origin.



Mathematical Focus 3

Verifying that f(-10)=3.873

The function represented by $f(x) = \left|\sqrt{x-5}\right|$ with domain and co-domain $\Re$ is a real function. Another way to express $f(x)$ is as the composition of two other functions, $g(x) = \sqrt{x-5}$, with domain $\Re$ and co-domain C, where C is the set of complex numbers, and $h(x) = |x|$ domain C and co-domain $\Re$. Since $f(x)$ is the composition of the functions represented by $g(x) = \sqrt{x-5}$ and $h(x) = |x|$, we can express $f(x)$ as $f(x) = h \circ g(x) = h(g(x))$. In this way, *f*(-10) has the same numerical value as *h*(*g*(-10)).

Since $-10 \in (-\infty, 5)$, $g : \Re \to C$, as shown below:

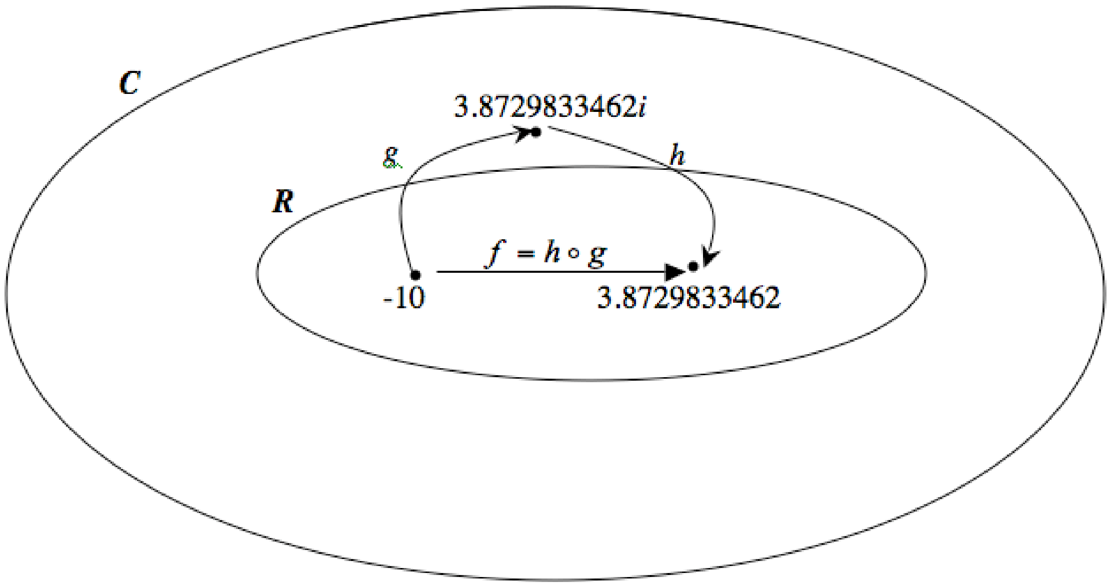
$g(-10) = \sqrt{-10-5} = \sqrt{-15} = 3.873i$

$3.873i \in C$, and $h : C \to \Re$, as shown below:

$h(g(-10)) = h(3.873i) = |3.873i| = 3.873$

Since $f(x) = h \circ g(x) = h(g(x))$, $f(-10) = 3.873$

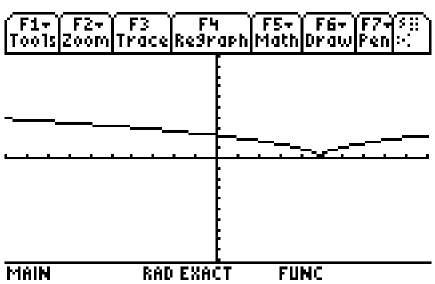
This progression can also be illustrated in the following diagram.



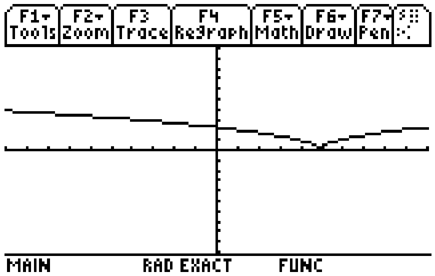
Mathematical Focus 4

Utilizing the concept of function as a mapping to make sense of f(-10)=3.873

The student-generated function in the prompt is $f(x) = \left|\sqrt{x-5}\right|$. Since the student was operating in complex mode on the CAS, the CAS returned a real value for $f(-10)$. We can consider the function represented by $f(x) = \left|\sqrt{x-5}\right|$ as a mapping from $\Re$ to $\Re$, i.e. $f:\left\{(x,y)\middle| x \in \Re \;\&\; y \in \Re \;\&\; y = \left|\sqrt{x-5}\right|\right\}$.



Any number of functions could describe the same set of ordered pairs as does $f:\left\{(x,y)\middle| x \in \Re \;\&\; y \in \Re \;\&\; y = \left|\sqrt{x-5}\right|\right\}$. For example, the mapping represented by $g:\left\{(x,y)\middle| x \in \Re \;\&\; y \in \Re \;\&\; y = \sqrt{|x-5|}\right\}$



describes the same set of ordered pairs as does the mapping represented by $f:\left\{(x,y)\middle| x\in\Re \;\&\; y\in\Re \;\&\; y=\left|\sqrt{x-5}\right|\right\}$. Therefore, since both functions describe the same set of ordered pairs, both functions will contain the ordered pair (-10, 3.873).