CAS Situation 5: Square root of *i*
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Prompt

Knowing that a CAS had commands such as **cfactor** and **csolve**, a teacher was curious about what would happen if she entered $\sqrt{i}$. The result was $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$. Why would a CAS give a result like this?

Commentary

When using a CAS, students and teachers can encounter situations that cause them to question why the CAS may give a particular result. Symbolic verification and manipulation can be used to confirm results given by a CAS. Mathematical focus 1 symbolically verifies $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ is a square root of i, and mathematical focus 2 uses symbolic manipulation to determine all square roots of i. Moreover, the CAS as a tool should be self-consistent in the results it produces, and mathematical focus 3 addresses how one can use the **csolve** command on a CAS to verify the tool's consistency related to $\sqrt{i}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

Mathematical foci 1, 2, and 3 account for the reasoning behind the symbolic work by confirming that the result makes sense. However, none of these foci deals with how $\sqrt{i}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ can make sense within a larger system. To address the underlying mathematical logic relating to why $\sqrt{i}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, mathematical foci 4, 5, and 6 utilize representations of complex numbers on the complex plane. Mathematical focus 4 connects powers of i to points of the unit circle on the complex plane and their images under rotations, and mathematical focus 5 considers the powers of i as elements of cyclic groups. Mathematical focus 6 uses Euler's formula to represent complex numbers in exponential and trigonometric form.

Mathematical Foci

Mathematical Focus 1

Verifying that $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ is a square root of i.

<u>Using Symbolic Verification</u>

One way to verify that a complex number is a square root of another number is to square that complex number and verify that the square and the other number are equivalent. By squaring the expression $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, we can verify that $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ is a square root of i. It is useful to note that the symbolic manipulations needed to expand the expression $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2$ will treat it as though it were an algebraic expression of the form $(a+b)^2$ from the real domain.

Expanding $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2$ gives $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2=\left(\frac{\sqrt{2}}{2}\right)^2+2\left(\frac{1}{2}\right)i+\left(\frac{\sqrt{2}}{2}i\right)^2$.

Simplifying $\left(\frac{\sqrt{2}}{2}\right)^2+2\left(\frac{1}{2}\right)i+\left(\frac{\sqrt{2}}{2}i\right)^2$ gives $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2=\frac{1}{2}+i-\frac{1}{2}=i$.

Since $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^2=i$, we conclude that a square root of i is equal to $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

Mathematical Focus 2

Solving the equation $x^2=i$, where $x=a+bi$

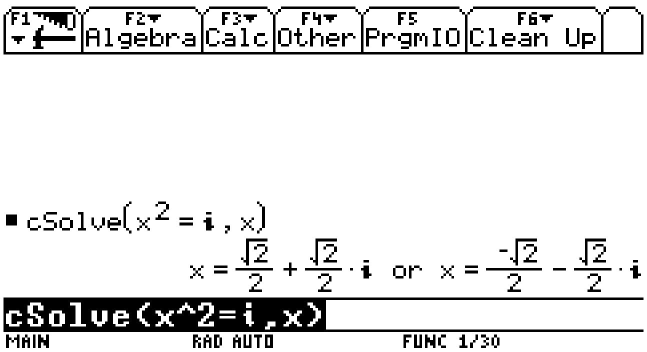
Knowing that any complex number is of the form $a+bi$, where a and b are real numbers, we can determine square roots of i by solving the equation $(a+bi)^2=i$ for a and b. To solve the equation, first we expand $(a+bi)^2$, and the equation becomes $a^2+2abi-b^2=i$. Equating the real and complex parts of the equation, $a^2-b^2=0$ and $2ab=1$. Therefore, $a=\pm b$ and either $2b^2=1$ or $-2b^2=1$. However, since we know that both a and b are real and that $-2b^2=1$ has no real solutions, we only consider the equation $2b^2=1$. However, if $a=-b$, then $2\cdot a\cdot b=2\cdot -b\cdot b=-2b^2=1$, which is not possible, meaning that $a=-b$ is not possible, leaving $a=b$ as the only possibility. Solving $2b^2=1$ for b gives $b=\frac{\sqrt{2}}{2}$ and $b=-\frac{\sqrt{2}}{2}$. Therefore the equation $(a+bi)^2=i$ has two sets of solutions, namely $a=\frac{\sqrt{2}}{2}$, $b=\frac{\sqrt{2}}{2}$ and $a=-\frac{\sqrt{2}}{2}$, $b=-\frac{\sqrt{2}}{2}$. In this way, $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ and $-\frac{\sqrt{2}}{2}+-\frac{\sqrt{2}}{2}i$ are both square roots of i.

Note: A very similar write-up of this focus can be found at the website http://www.math.toronto.edu/mathnet/questionCorner/rootofi.html.

Mathematical Focus 3

Using the CAS to solve the equation $x^2=i$, where $x=a+bi$

As a tool, the CAS should be self-consistent, in that if it provides the result $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ when the user enters $\sqrt{i}$, then when solving an equation with solution $x=\sqrt{i}$, the CAS should provide the result $x=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$. By using the command **cSolve** on the TI-92 Plus, as shown in the screen shot below, we can solve the equation $x^2=i$ for the complex variable x.



Mathematical Focus 4
Relating powers of *i* to rotations involving the unit circle on the complex plane

Consider the unit circle on the complex plane, and on this circle, consider the point representations of i^0 and i (**figure 1**).

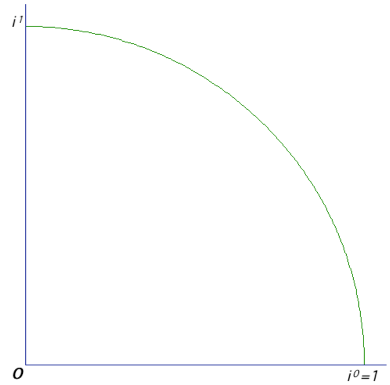


Figure 1 First quadrant of the unit circle on the complex plane.

Note that the point representing i is the image of the point representing i^0 under $\rho_{(O,90°)}$, a rotation of 90° about the origin (***O***). Thus, if the point for i^0 could be represented as (1,0), and if the point for i could be represented as $\rho_{(O,90°)}((1,0))=(0,1)$, then the point for $\sqrt{i}$ can be thought of as the image of the point for i^0 under $\rho_{(O,45°)}$, a rotation of 45° (**figure 2**). Moreover, the point for i can also be thought about as the image of the point for $\sqrt{i}$ under $\rho_{(O,45°)}$ So, $\rho_{(O,45°)}$ composed with itself is the same as $\rho_{(O,90°)}$. That is, $\rho^2_{(O,45°)}.=\rho_{(O,90°)}$

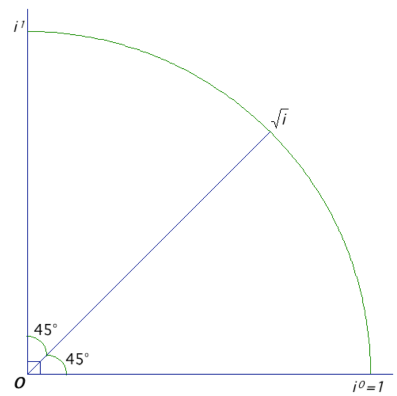


Figure 2 Images of points representing powers of *i* as rotations.

We can notice that each point on the circle corresponds to the complex number, $\cos x + i\sin x$. This is shown in **figure 3**.

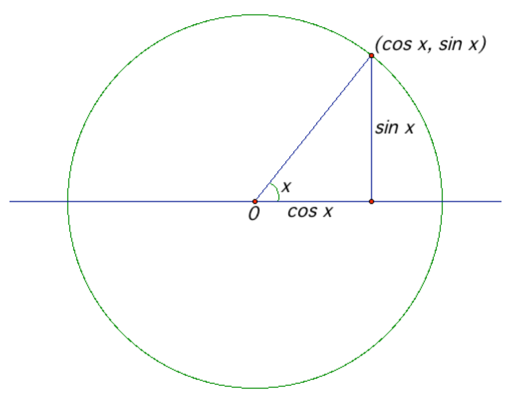


Figure 3 Coordinates of Points on the Complex Unit Circle

The point representing $i^{\frac{1}{2}}$ is the image of the point for i^0 under a rotation of 45° about the origin. Therefore, the coordinates of $i^{\frac{1}{2}}$ have to be $(\cos 45°, \sin 45°) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Thus, $i^{\frac{1}{2}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

Mathematical Focus 5

Considering cyclic groups

This situation deals with $\sqrt{i}$, which can of course be written as $i^{\frac{1}{2}}$. So, one way to go about this situation is to look for patterns in the powers of *i*. To begin with, let's look at the integer powers of *i*, starting with i^0. If we were to plot points representing the imaginary numbers i^0, i^1, i^2, i^3, we obtain the following figure (**figure 4**).

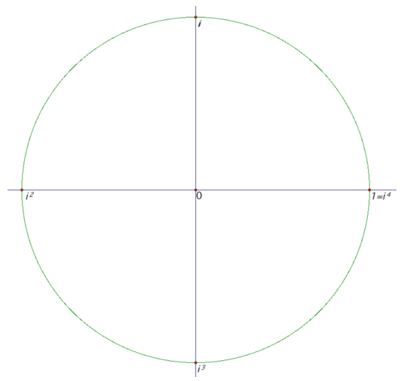


Figure 4 The four powers of *i*

Note that all four powers of *i* above are on the complex unit circle. Moreover, the four points are positioned at equal increments around the circle (exactly at 90°increments). Furthermore, we can see the that fourth power of *i* can be plotted in the same position as the zero power of *i* (i.e., $i^4 = i^0 = 1$). We can also see that every integer power of *i* greater than 3 is plotted on the above four points around the complex unit circle. This arrangement, at equal increments, around a circle of the four powers of *i*, and the cyclic property of the powers described above, leads to looking at a cyclic group generated by *i*.

Consider a cyclic group, $(G,\circ)$, of order 4, isomorphic to ($\mathbf{Z}_4$,+), which can be generated by using the imaginary number *i* as the generator, i.e. $i^{4k} = 1$, where $k \geq 0$ and *k* is an integer. Note that 1 is called the identity element of the group *G*. Also, we can list all the elements of this group by considering the powers of *i*, i.e. $G = \langle i \rangle_4 = \{1, i^1, i^2, i^3\}$. As discussed before, the elements of the cyclic group, *G*, can be very naturally illustrated as four symmetric points on the unit circle in the complex plane, as shown in **figure 4**.

However, since we are interested in $i^{\frac{1}{2}}$, we can further this discussion of the powers of *i*, by examining the first eight powers of *i*, increasing the powers in increments of $\frac{1}{2}$. Thus, now we are increasing the order of the group from 4 to 8. So now we have the cyclic group, $(H,\circ)$, isomorphic to ($\mathbf{Z}_8$,+), given by $H = \langle i \rangle_8 = \{1, i^{\frac{1}{2}}, i^1, i^{\frac{3}{2}}, i^2, i^{\frac{5}{2}}, i^3, i^{\frac{7}{2}}\}$. This group can also be illustrated on the unit circle on the complex plane as shown in **figure 5**.

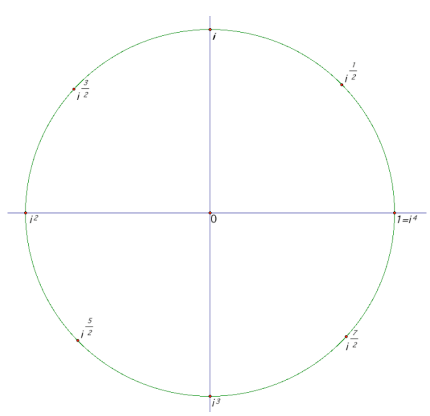


Figure 5 The Cyclic Group $(H, \circ)$

To obtain the co-ordinates of these points on the complex unit circle, we can refer back to focus 3 and obtain $i^{1/2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

Mathematical Focus 6

Appealing to Euler's formula and the geometry of the complex plane

Knowing that every point on the unit circle on the complex plane corresponds to a complex number z, where $z = \cos\theta + i\sin\theta$, Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, can be used to express those complex numbers in the exponential form $z = e^{i\theta}$. For example, if we let $\theta = \pi$, we arrive at $z = e^{i\pi} = \cos\pi + i\sin\pi = -1$, which can be represented by the point (-1,0) on the unit circle on the complex plane. Similarly, if we let $\theta = \frac{\pi}{2}$, we arrive at $z = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$, which can be represented by the point (0,1) on the unit circle on the complex plane. Since we are interested in determining $\sqrt{i}$ and since $e^{i\frac{\pi}{2}} = i$, by Euler's formula, it follows that $\sqrt{e^{i\frac{\pi}{2}}} = \sqrt{i}$. Since $i^{\frac{1}{2}} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}}$, using properties of exponents, we can conclude that $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$. In this way, if we let $\theta = \frac{\pi}{4}$, we arrive at $z = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, which can be represented by the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle on the complex plane. Since $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, and $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$, we can conclude that $i^{\frac{1}{2}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

2nd Commentary (Reflection)

There are important ideas that connect representations and raise the issues of knowing which idea(s) to use and when to use the idea(s). By calling on trigonometry when seeing $\frac{\sqrt{2}}{2}$, the expression $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ becomes a specific case of $\cos x + i\sin x$, where $x = \frac{\pi}{4}$. This leads to the notion of the unit circle on the complex plane and the implications that would have, as the complex number $\cos x + i\sin x$ has coordinates $(\cos x, \sin x)$ on the complex plane. Euler's Formula connects Taylor Polynomials and derivatives to the complex unit circle. These sets of ideas are each well connected and each produce results, and we want teachers to be able to draw on them when they are useful. By having this ability or intuition, secondary teachers increase their effectiveness of coming up with a plausible and understandable explanation of the student's question.

Considering the concept of exponentiation as rotation rather than just as repeated multiplication leads one to the question of what exponentiation really means? Is it only when we get to complex numbers that a conception of exponentiation as rotation makes sense? What about the interpretation of exponentiation as dimension? Is it a natural thing to start by introducing exponentiation as a direction to multiply something by itself? Or, if exponentiation is naturally introduced as repeated multiplication, how would these other conceptions come about?

References:

http://mathworld.wolfram.com/i.html
http://www.math.toronto.edu/mathnet/questionCorner/rootofi.html