MAC-CPTM Situations Project

Situation 07: Temperature Conversion

Prepared at Pennsylvania State University
Mid-Atlantic Center for Mathematics Teaching and Learning
9 June 2005 – Glen Blume

Edited at Pennsylvania State University
Mid-Atlantic Center for Mathematics Teaching and Learning
21 January 2009 – Heather Godine, Maureen Grady, and Svetlana Konnova
07 September 2009 – M. Kathleen Heid

Prompt

During a high school first-year Algebra class, students were given the task of coming up with a formula that would convert Celsius temperatures to Fahrenheit temperatures, given that in Celsius 0° is the temperature at which water freezes and 100° is the temperature at which water boils, and given that in Fahrenheit 32° is the temperature at which water freezes and 212° is the temperature at which water boils.
The rationale for the task is that if one encounters a relatively unfamiliar Celsius temperature, one could use this formula to convert to an equivalent, perhaps more familiar in the United States, Fahrenheit temperature (or vice versa).
One student developed a formula based on reasoning about the known values from the two temperature scales.

"Since 0 and 100 are the two values I know on the Celsius scale and 32 and 212 are the ones I know on the Fahrenheit scale, I can plot the points (0, 100) and (32, 212). If I have two points I can find the equation of the line passing through those two points.

(0, 100) means that the y-intercept is 100. The change in y is (212-100) over the change in x, (32-0), so the slope is $\frac{112}{32}$. Since $\frac{112}{32} = \frac{7*16}{2*16}$, if I cancel the 16s the slope is $\frac{7}{2}$. So the formula is $y = \frac{7}{2}x + 100$."

Commentary

The situation addresses several key concepts in school mathematics that relate to temperature scales and conversion between scales. The first three foci highlight the concepts of interval scale, composition of two functions, function, and domain and range of a function. The next three foci use geometrical and graphical approaches to convert between scales, appealing to the concept of the inverse function and composition of three functions. This Situation is related to Situation 03, Inverse Trigonometric Functions.

Mathematical Foci

Mathematical Focus 1

Celsius and Fahrenheit temperature scales are interval scales. Therefore, one can use linear transformations to move between them.

Choosing two reference temperatures and dividing the temperature difference between these two points into a certain number of degrees determine a temperature scale. The temperature scale is an interval scale, because the difference between two scale values can be meaningfully compared with the difference between two other scale values and the differences between numbers reflect the differences among properties in the real world. Because the units on the scale are constant, it is always true that, for example, the difference between 40° and 30° is the same as the differences between 30° and 20°. However, it is not, in general, meaningful to say that 40° is twice as hot as 20°. The reason is that the zero degree on the temperature scales (except the Kelvin temperature scale) does not represent zero amount of heat.

The two reference temperatures used for most common scales are the melting point of ice and the boiling point of water. On the Celsius temperature scale, the melting point is taken as 0°C and the boiling point as 100°C, and the difference between them is partitioned into 100 degrees. On the Fahrenheit temperature scale, the melting point is taken as 32°F and the boiling point as 212°F, and the difference between them is partitioned into 180 degrees. The zero point on the Celsius or Fahrenheit temperature scales are arbitrary. For example, the zero point on the Celsius scale has been arbitrarily set at the freezing point of water.

In an interval scale, the relative difference among scale values is unaffected by any linear transformation of the form

$$y = ax + b \ , \qquad (1)$$

where x is the original scale value, and y is the transformed scale value.

The value of the multiplier constant determines the arbitrary size of the scale unit, and the value of the additive constant b determines the arbitrary location of the zero point on the scale. In other words, the results after transformation retain the same meaning and refer to the same phenomena as the original scale value. For example, through the linear transformation (1) we can move from the 0°C to the 32° F and both values will refer to the same phenomena of the freezing point of water.

Mathematical Focus 2

To convert from Fahrenheit to Celsius temperature scales, one can apply the concept of composition of the functions.

Consider the following diagram, which represents Celsius, Intermediate and Fahrenheit scales.

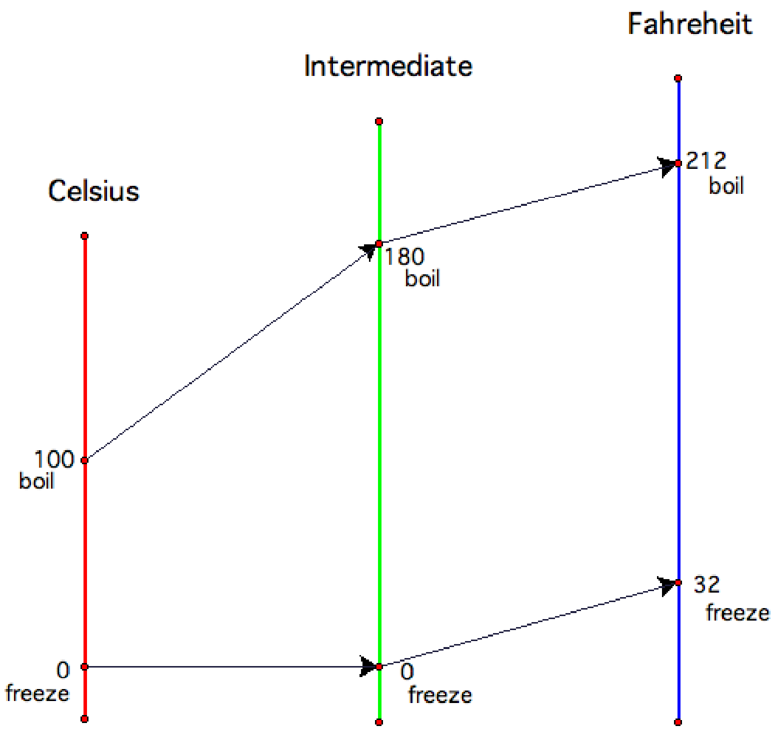

Figure 1

To make the conversion possible, two features need to match. There needs to be an equal number of degrees between the freezing and boiling points, and the freezing and boiling points have to match. (*See CAS-IM Module II (Heid & Zbiek, 2004) exploration of composition of functions via problems involving scale conversions*).

The first step is to map the Celsius scale to the intermediate scale that is comparable to the distance between the freezing point and the boiling point on the Fahrenheit scale. On the Celsius scale the distance between the freezing and boiling points is 100, since 100-0=100. On the Fahrenheit scale the distance between the freezing and boil points is 180, since 212-32=180. Therefore, for every 100 degrees on the Celsius scale there are 180 degrees on the intermediate scale. Since $\frac{180}{100} = \frac{9}{5}$, the scale factor should be $\frac{9}{5}$. It means that for every 9 degrees on the intermediate scale there are 5 degrees on the Celsius scale. Consequently, to make the number of degrees between the freezing and boiling points on the intermediate scale equal to the number of degrees between the freezing and boiling points on the Fahrenheit scale, one needs to multiply the Celsius-scale temperatures by $\frac{9}{5}$. This first step corresponds to the function g, that takes the temperature in Celsius (w) as the input and multiplies it by the scale factor $\frac{9}{5}$.

$g(w) = \frac{9}{5}w$

The second step is to get the freezing points to match. Since, the freezing point on the Celsius scale is 0° and the freezing point on the Fahrenheit scale is 32°, then to get both freezing points to match, one has to add 32° to each of the intermediate-scale temperatures. The freezing temperature moves up from 0° on the intermediate scale to 32° on the Fahrenheit scale, and the boiling point moves to 212° on the Fahrenheit scale. This second step corresponds to the function h, that takes $g(w)$, the output of the function g as the input and adds 32°.

So $h(r) = r + 32$ for $r = g(w)$.

The two function rules can be written as follows:

$g(w) = \frac{9}{5}w$ and $h(r) = r + 32$

As a result, the function f that converts Celsius temperatures to Fahrenheit temperatures is the composition of the function g followed by the function h.

$f = h \circ g$, where $g(w) = \frac{9}{5}w$ and $h(r) = r + 32$

Mathematical Focus 3

The relation that associates the Celsius and Fahrenheit temperature scales is a function.

The student was asked to write the formula that would convert Celsius temperatures to Fahrenheit temperatures. This means the student had to determine the relation that associates with each element of X (the Celsius temperature value) exactly one element of Y (the Fahrenheit temperature value) and write the corresponding equation. Therefore, by definition, this relation is a function. The set X is called the domain of the function. The set Y of all images of the elements of the domain is called the range of the function.

According to the prompt, two scales represent the same phenomena (water freezes and water boils). We can use these facts to determine the relations between values of the temperature given in Celsius and values of the temperature given in Fahrenheit.

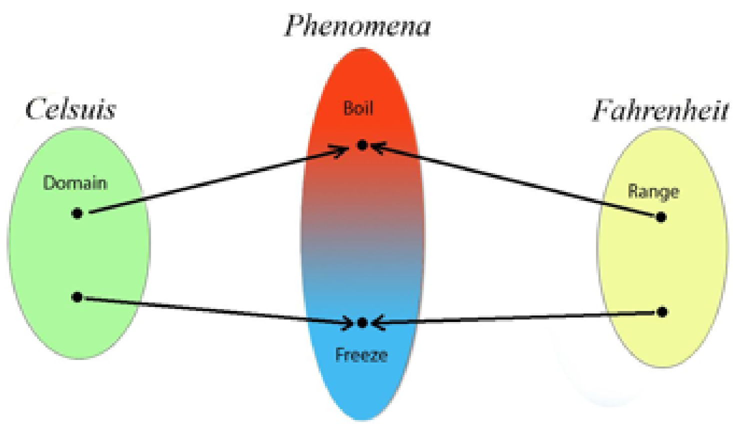

Figure 2

The domain of the function consists of the Celsius temperature values and the range of the function consists of the Fahrenheit temperature values. Therefore, $0^o$and 100^o belong to the domain of the function and 32^o and 212^o belong to the range of the function.

The equation used for the conversion from the Celsius temperature scale to the Fahrenheit temperature scale is a linear function (see Focus 1). Therefore, we need two "points" with the coordinates (x_1, y_1) and (x_2, y_2) to find the equation for that line.

	Celsius temperature values	Fahrenheit temperature values
	X	Y
Boiling point	0^0	32^0
Freezing point	100^0	212^0

The first point corresponds to the phenomena of boiling water. Therefore, the value of the independent variable, x_1, belongs to the Celsius temperature values, so $x_1 = 0$, and the value of the dependent variable, y_1, belongs to the Fahrenheit temperature values, so $y_1 = 32^0$.

The second point corresponds to the phenomenon of freezing water. Therefore, the value of the independent variable, x_2 is 100^0, and the value of the dependent variable, y_2, is 212^0..

One can write an equation in slope-intercept form:

$$y - y_1 = m(x - x_1) \qquad (2)$$

for the line through two points $(0,32)$ and $(100,212)$.
First, find the slope of the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}.$$

Second, substitute the slope and one of the points $(0,32)$, rewrite and simplify the equation (2)

$$y - 32 = \frac{9}{5}(x - 0)$$

$$y = \frac{9}{5}x + 32 \qquad (3)$$

The last equation (3) is the equation that converts the Celsius temperature values to the Fahrenheit temperature values.

Mathematical Focus 4

Using a graphical- geometric approach to create the formula to convert Celsius temperature to Fahrenheit temperature.

M Kathleen Heid 9/7/09 5:28 PM
Comment: I am not convinced of the value of this focus. I suggest omitting it.

To find the formula to convert the temperature values in the Celsius temperature scale to temperature values in the Fahrenheit temperature scale, one can use coordinate geometry. In this case we know the corresponding temperature values in the both scales.

f and c – the temperature values in the Fahrenheit and Celsius temperature scales, corresponding to one arbitrary chosen phenomenon

$f_1 = 32^0$ – the temperature value in the Fahrenheit temperature scale, when water is freezing

$f_2 = 212^0$ – the temperature value in the Fahrenheit temperature scale, when water is boiling

$c_1 = 0^0$ – the temperature value in the Celsius temperature scale, when water is freezing

$c_2 = 100^0$ – the temperature value in the Celsius temperature scale, when water is boiling

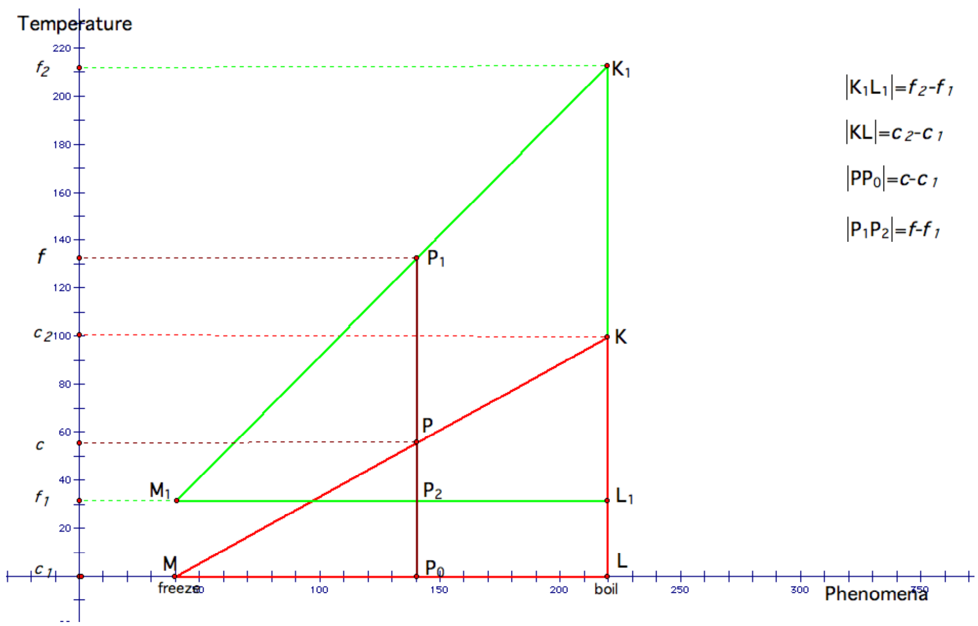

Figure 3

The segment *MK* represents the Celsius temperature scale, and the segment M_1K_1 represents the Fahrenheit temperature scale (see Figure 3).

Choose any point P_0 between points *M* and *L* on the Phenomena- axis and draw the line parallel to the Temperature axis. This line intersects the segment *MK* at point *P*, and segment M_1K_1 at point P_1. Therefore, the temperature coordinate of the point *P* represents the Celsius temperature value and the temperature coordinate of the point P_1 represent the corresponding Fahrenheit temperature value.

Let consider two pairs of the triangles $\triangle LMK$ and $\triangle P_0MP$, and $\triangle L_1M_1K_1$ and $\triangle P_2M_1P_1$. We can apply the Parallel Side-Splitter Theorem to prove that

$$\frac{|PP_0|}{|KL|} = \frac{|MP_0|}{|ML|} \text{ and } \frac{|P_1P_2|}{|K_1L_1|} = \frac{|M_1P_2|}{|M_1L_1|} = \frac{|MP_0|}{|ML|},$$

Therefore,

$$\frac{|P_1P_2|}{|K_1L_1|} = \frac{|PP_0|}{|KL|}$$

$$\frac{f_2 - f_1}{f - f_1} = \frac{c_2 - c_1}{c - c_1}$$

$$f = c\frac{f_2 - f_1}{c_2 - c_1} - c_1\frac{f_2 - f_1}{c_2 - c_1} + f_1$$

Using substitution of the values c_1, c_2, f_1, f_2, we can rewrite and simplify the equation

$$f = c\frac{212 - 32}{100 - 0} - 0\frac{212 - 32}{100 - 0} + 32$$

$$f = \frac{9}{5}c + 32$$

Mathematical Focus 5

By applying the rule for converting from the Celsius scale to the Fahrenheit scale and applying the concept of the inverse function, one can create the rule for converting from Fahrenheit scale to Celsius scale

Let *f* be a function whose domain is the set X, and whose range is the set Y. Then, if it exists, the inverse of *f* is the function f^{-1} with domain Y and range X, defined by the following rule:

If $f(x) = y$, then $f^{-1}(y) = x$

In case of interval scale functions the transformation between them is a one- to–one correspondence. Therefore, for each linear function we can find the inverse function.

For example, let *f* be the function that converts a temperature in degrees Celsius (c) to a temperature in degrees Fahrenheit *f(c)* (see Focus 3):

$$f = \frac{9}{5}c + 32$$

One approach to finding a formula for f^{-1} is to solve the equation $y=f(c)$ for *c*.

$$c = \frac{5}{9}(f - 32) \qquad (4)$$

Thus the inverse function converts Fahrenheit degrees to Celsius degrees:

$$f^{-}(f) = \frac{5}{9}(f - 32)$$

$$c(f) = \frac{5}{9}(f - 32)$$

Mathematical Focus 6

Converting between scales by applying compositions of functions

The Kelvin scale is a third standard temperature scale. It is the temperature scale that can be defined theoretically, for which zero degree (0^{o}) corresponds to zero average kinetic energy. Scientists have determined that the coldest it can get, in theory, is minus 273^{o} degrees Celsius. The Kelvin temperature scale is an absolute scale having degrees the same size as those of the Celsius temperature scale.

To write the function rule to convert Celsius temperatures to Kelvin temperatures, the scale factor has to be determined and the freezing points have to match (see Mathematical Focus 1).

The scale factor of the transformation between the Celsius temperature scale and the Kelvin temperature scale should be equal 1 (as long as they have the same size of the degrees). On the Celsius scale water freezes at 0^{o} and on the Kelvin scale water freezes at 273^{o}. Therefore, the function, k, that takes temperature in degrees Celsius, c, to a temperature in degrees Kelvin, has the rule

$$k(c) = c + 273$$

To find the composition of the functions that will convert from Fahrenheit scale to Kelvin scale, we can use the function rule that converts Fahrenheit to Celsius that was established in Mathematical Focus 5

$$c(f) = \frac{5}{9}(f - 32)$$

and compose the last two functions:

if $c(f) = \frac{5}{9}(f - 32)$, and $k(c) = c + 273$, then the composition

$$k(c(f) = \frac{5}{9}(f - 32) + 273$$

is a composition that will convert Fahrenheit to Kelvin.

References

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Pedhazur, E. J., & Schmelkin, L. P. (1991). Measurement, Design, and Analysis: An Integrated Approach. Lawrence Erlbaum Associates.