Geometric Proof

In this essay, we look at the Euclidean approach to the mathematical proof of geometrical facts. We will start with basic facts called axioms, or from other previously proven facts. Using these we will establish a chain of reasoning that demonstrate the truth of a particular statement or proposition.

The first basic assumption we will make is that a full turn is 360°.

Theorem 1 Angles at a point add to 360°

Proof

In the GSP sketch, the angles make up a full turn, and a full turn is 360°, so

\[ a + b + c + d + e + f = 360° \]

This argument would hold for any number of angles at a given point. We have illustrated it for six angles.

Example
Definition: A corollary is a fact that results from a significant theorem.

We can use Theorem 1, and the fact that angles on either side of a straight line are equal, to deduce a.

**Corollary 1: The angle on a straight line is 180°.**

**Proof:**

In the GSP sketch, a and b are angles at a given point (common vertex) so that \( a + b = 360° \), by Theorem 1.

But \( a = b \), so \( a + a = 360 \) degrees or \( 2a = 360 \) degrees, which gives \( a = 180° \). QED.
Example

**Corollary 2: The sum of angles on a straight line is 180°**

![Diagram of angles on a straight line](image1)

**Proof:**

In the diagram above, a and b are angles on a straight line, so $a + b = 180°$ by corollary 1.

**Theorem 2: Vertically opposite angles are equal**

![Diagram of vertically opposite angles](image2)

**Proof:**

In the diagram above, angles a and b make up a straight line.
So, \( a + b = 180^\circ \), by corollary 2. Angles \( a \) and \( c \) also make up a straight line.

\[ a + c = 180^\circ, \text{ by corollary 2} \]

Therefore \( a + b = a + c \)

Subtracting \( a \) from both sides of the equation yield, \( b = c \).

The vertical opposite angles \( b \) and \( c \) are equal.

Similarly,

\[ a + b = 180^\circ \quad \text{(angles on a straight line)} \]
\[ d + b = 180^\circ \quad \text{(angles on a straight line)} \]

Therefore, \( a + b = d + b \)

Subtracting \( b \) from both sides we get, \( a = d \). QED.

**Parallel Lines**

Definition: Parallel lines are lines that are always equal distance apart. Parallel lines never meet.

**Euclid’s 5th Axiom**

If a straight line \( XY \) meets two other straight lines \( LM \) and \( PQ \) (see diagram below) so that \( a + b \) is not \( = \) to \( 180^\circ \), then \( LM \) and \( PQ \) will meet, that is line \( LM \) and line \( PQ \) are not parallel.
Theorem 3: If LM and PQ are parallel lines intersected by a third line (transversal) XY, then alternate interior angles are equal. That is: angle $a = c$ and angle $b = d$.

Proof:
In the diagram above, the fact that LM and PQ are parallel implies that,

\[ a + b = 180 \]  \hspace{1cm} \text{(by Euclid's 5th axiom)}

However, c and d are angles on a straight line, so

\[ c + b = 180^\circ \]  \hspace{1cm} \text{(by corollary 2)}

Therefore, \( a + b = c + b \)

Subtracting b from both sides of the equation yields

\[ a = c. \text{ QED.} \]

Example

\[
\begin{align*}
\angle FCD &= 112.4^\circ \\
\angle CAG &= 112.4^\circ \\
\angle DCA &= 67.6^\circ \\
\angle GAH &= 67.6^\circ \\
\angle ECF &= 67.6^\circ \\
\angle BAC &= 67.6^\circ \\
\angle ACE &= 112.4^\circ \\
\angle HAB &= 112.4^\circ \\
\end{align*}
\]

Theorem 4: The sum of the angles of every triangle is 180°

That is, \( a + b + c = 180^\circ \)
Theorem 4

Proof

In the sketch above we extend the line segment BC to U and draw line segment CW parallel to AB. We have

- \( a = d \)
- \( c = e \)

But \( b + d + e = 180^\circ \)

Therefore, \( a + b + c = 180^\circ \) QED.

\[
\begin{align*}
m\angle ABC &= 50.63^\circ \\
m\angle CAB &= 99.71^\circ \\
m\angle ACB &= 29.66^\circ \\
m\angle ABC + m\angle CAB + m\angle ACB &= 180.00^\circ
\end{align*}
\]