Cost of Fencing for a Field
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Goal: To find the minimum cost of fencing given different costs

Problem

A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs $2 per foot, while the fence for the other three sides costs $1 per foot. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is the minimum expense?

Use the AM-GM Inequality to find a solution. No calculus needed.

AM GM solution

Let side of fence facing road (say south) be x and so north is x, east and west sides are y

Side x to the road costs 2 per foot, side y costs 1 per foot.
\[ 2x + x + y + y = \text{cost} \quad 3x + 2y = \text{cost} \quad [1] \]
\[ x \times y = 60,000 \text{ so } y = \frac{60000}{x} \quad [2] \]


\[ 3x + \frac{120000}{x} \text{ is cost function} \]

Graph

\[ Y = 3x + 2 \left( \frac{6000}{x} \right) \]

Compare to

\[ Y = 2 \sqrt{(3x)(2)\frac{6000}{x}} \]
\[ 2 \sqrt{(3^2)(2^2)(100^2)} \]
\[ = 2 \times 3 \times 2 \times 100 \]
\[ = 1200 \]
\[ \text{AM = GM} \]
\[ \text{iff } 3x = 2 \left( \frac{6000}{x} \right) \]
\[ x^2 = 40000 \]
\[ x = 200 \]

Now if one side is 200 we show the other side is 300 = 60000/200

Minimum Cost = 3(200) + 2 \left( \frac{60000}{200} \right) = 600 + 600 = $1200

Solution using Calculus
Using $x$ as the side facing the road as in the above we have

$3x + 120000/x$ is cost function

Derivative is $3 - 120000/x^2$

Set $= 0$ to minimize

$3x^2 = 120000$

$x^2 = 40000$

$x = 200$

Cost $= 3x + 120000/x = 3(200) + 120000/200 = 1200$

The fencing uses more of the cheaper material to offset to costs of the more expensive side..