Situation: What is π?

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Prompt

During a lesson on trigonometric functions in a pre-calculus class, students were learning to extend the domain of trigonometric functions using the unit circle. As students were examining the values of sine, cosine and tangent for $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$, a student raised their hand and questioned, "What exactly is π?"

Commentary

The constant π is a real number defined as the ratio of a circle's circumference to its diameter. This famous number is one that dates back to ancient civilizations such as the Babylonians and Egyptians. In this time, the use of the number was used in order to evaluate the area or perimeter of circular fields. Its value was one that was implicit and not yet defined as a constant. Due to the irrationality of π, it can never be expressed exactly as the ratio of any two integers. It thus follows that the decimal representation of π is never ending, i.e., does not have a repetend. Since this is the case, numerous approximations for π have been used throughout history. With modern technology today, the decimal digits of π have been calculated to very extensive lengths, but because there are infinitely many decimal digits of π, no exact value will be known. Listed below are some approximations historically used ending with a more modern approximation. This list is by no means exhaustive.

Author	Year	Exact Digits	Method	Comment
Egyptians	2000 B.C.	1	unknown	$\pi = \left({}^{16}/_{9}\right)^2$
Babylonians	2000 B.C.	1	unknown	$\pi = 3 + \left({}^{1}/_{8}\right)$
Bible	550 B.C. ?	0	unknown	$\pi = 3$
Archimedes	250 B.C.	2	polygon	$\pi = {}^{22}/_{7}$, 96 sides
Brahmagupta	640	1	unknown	$\pi = \sqrt{10}$
Fibonacci	1220	3		$\pi = 3.141818$
Yee and Kondo	2010	55 trillion	Series	PC Intel Xeon

The series used by Yee and Kondo is as follows:

$$\frac{1}{\pi} = 12 \sum_{k-0}^{\infty} (-1)^k \frac{(6k)!}{(3k)!\,(k!)^3} \frac{(13591409 + 545140134k)}{640320^{3k+3/2}}$$

This set of foci in this situation aim to provide a developmental overview of some of the crucial properties associated with the well-known number π with some further historical perspective associated with their development.

Mathematical Foci

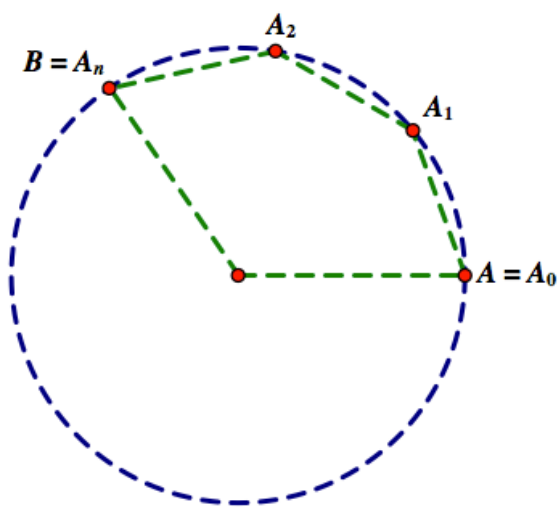
Mathematical Focus 1

In order to prove the existence of the number π, it needs to be shown that the ratio of the circumference to the diameter is the same for all circles.

Prior to proving the existence of the number π as the ratio of the circumference of any circle to its diameter, we must first examine some background information that will allow us to show this number is the same for all circles.

The Definition of Arc Length

Given an arc $\widehat{AB}$ of a circle C:



We take a sequence of points

$$A = A_0, A_1, A_2, \dots, A_n = B,$$

in the order from A to B on the arc; and for each pair of successive points A_{i-1}, A_i we draw the segment $\overline{A_{i-1}A_i}$, as shown in the figure above. The union of these segments is called an inscribed broken line and the sum of their lengths is denoted by p_n. Thus

$$p_n = A_0A_1 + A_1A_2 + \cdots + A_{n-1}A_n = \sum_{i=1}^{n} A_{i-1}A_i$$

There are now various ways that we might define the length of the arc $\widehat{AB}$. If we merely want to state a definition, as a matter of form, without intending ever to apply it, then our problem is simple. We agree to use equally spaced points $A_0, A_1, A_2, \dots, A_n$. The length p_n of our broken line is now completely determined by n, and we can define the length to be

$$p = \lim_{n\to\infty} p_n.$$

To justify this, we would need to explain what is meant by $lim_{n\to\infty}$, and we would have to show that the indicated limit really exists, for every circular arc.

The following definition, however, is far more manageable. Let P be the set of all numbers p_n, which are lengths of broken lines inscribed in $\widehat{AB}$. Thus

$$P = \left\{p_n \mid p_n = \sum A_{i-1}A_i\right\}.$$

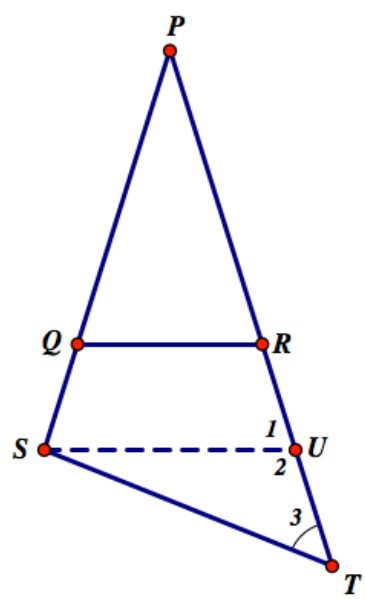
Let

$$p = \sup P.$$

To justify this, we need to prove the following theorem.

Theorem. P has an upper bound.

It will follow, of course, that P has a *least* upper bound sup P. The proof is easy on the basis of the following preliminary result. Let $\triangle\, PQR$ be an isosceles triangle, with $PQ = PR$.



We assert that if P-Q-S and P-R-T, then

$$ST > QR.$$

Suppose that $PS < PT$, as in the figure, and take U between R and T so that $\overline{SU} || \overline{QR}$. Then

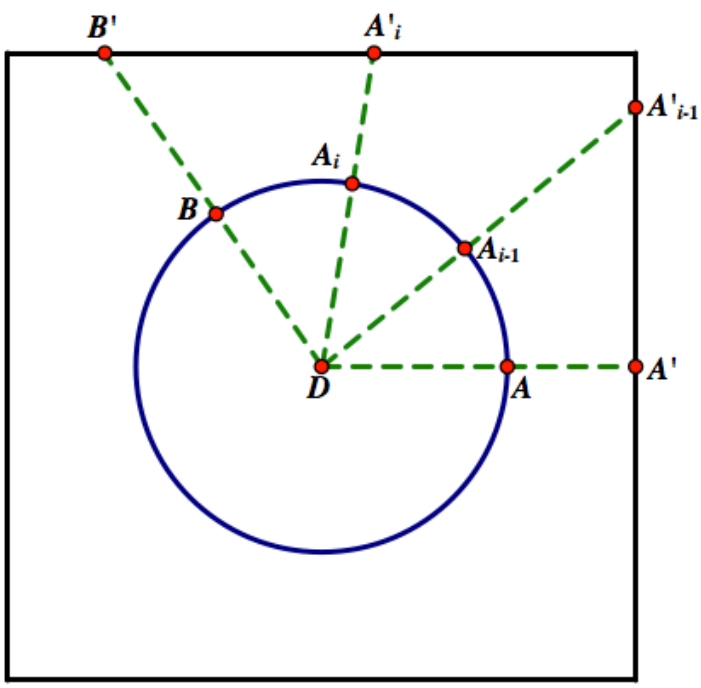
$$\frac{SU}{QR} = \frac{PS}{PQ} > 1.$$

Therefore

$$SU > QR.$$

We shall now show that $ST > SU$. Evidently $\angle 1$ is acute, because $\angle 1$ is a base angle of and isosceles triangle. Therefore $\angle 2$ is obtuse. Therefore $\angle 3$ is acute. Therefore $m\angle 3 < m\angle 2$. Therefore $ST > SU$.

We now return to our circular arc. Draw any square that contains the whole circle in its interior, as shown below.



We project each point A_i onto the square, as indicated in the figure. That is, A'_i is the point where $\overrightarrow{DA_i}$ intersects the square. Then $A_{i-1}A_i < A'_{i-1}A'_i$. Therefore p_n is always less than $\sum_{i=1}^{n} A_{i-1}A_i$. Therefore p_n is always less than the perimeter of the square. Thus the perimeter of the square is the upper bound that we are looking for. This justifies our definition

$$p = \sup P.$$

Of course a circle is not an arc of itself, under our definition of an arc. But we can define the circumference of a circle in an entirely analogous way, by setting up an inscribed polygon with vertices

$$A_0, A_1, \dots, A_{n-1}, A_n = A_0$$

We then let p_n be the perimeter

$$\sum_{i=1}^{n} A_{i-1}A_i,$$

Let P be the set of all such perimeters p_n, and define the circumference as

$$p = \sup P.$$

The Existence of π

We now want to prove the existence of the number π. To do this, we need to show that the ratio of the circumference to the diameter is the same for all circles. This is a theorem about sup's, and to prove it, we need a preliminary result on sup's.

Let P be any bounded set of positive numbers, and let k be a positive number. Then kP denotes the set of all numbers of the form kp, where p belongs to P. For example, if

$$p = [0,1] = \{x | 0 \leqq x \leqq 1\},$$

and $k = 3$, then

$$kP = 3P = [0,3].$$

If $P = [1,2]$ and $k = \frac{2}{3}$, then $kP = \left[\frac{2}{3}, \frac{4}{3}\right]$; and so on.

This "multiplication" is associative. That is,

$$j(kP) = (jk)P,$$

because

$$\{jx | x \in kP\} = \{j(kp) | p \in P\} = \{(jk)p | p \in P\}.$$

Thus, for example, we always have

$$\frac{1}{k}(kP) = P.$$

Lemma 1. If b is an upper bound of P, then kb is an upper bound of kP.
Reason. If $p \leqq b$, then $kp \leqq kb$.

Lemma 2. If c is an upper bound of kP, then c/k is an upper bound of P.
Proof. Since $P = \left({}^{1}/_{k}\right)(kP)$, this follows Lemma 1.

These lemmas give us the following theorem.

Theorem. $\sup(kP) = k \sup P$.
Proof. Let $b = \sup P$. Then b is an upper bound of P. By Lemma 1, kb is an upper bound of kP.

Suppose that kP has an upper bound

$$c < kb.$$

Then $\frac{c}{k}$ is an upper bound of P, by Lemma 2. This is impossible, because $\frac{c}{k} < b$, and b was the least upper bound of P.

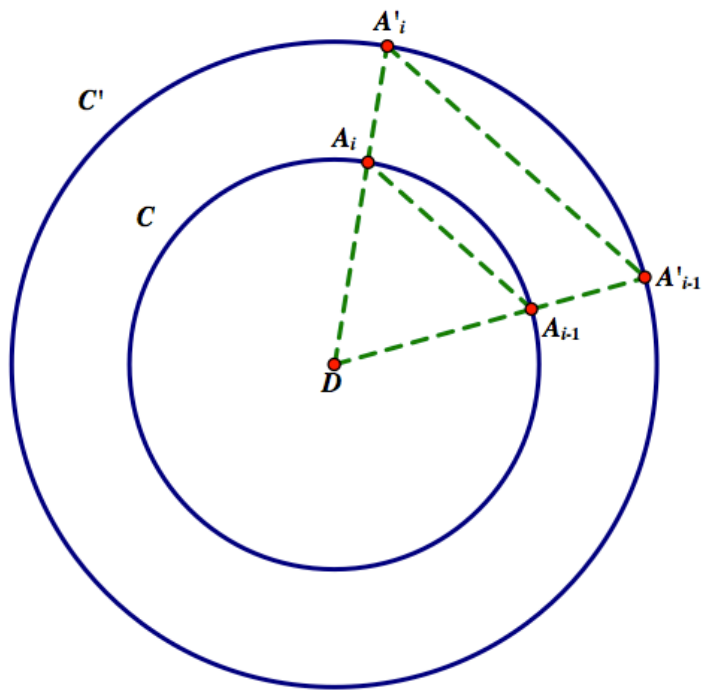
We can now prove the theorem, which establishes the existence of π. What is needed is the following theorem.

Theorem. Let C and C' be circles with radii r, r' and circumferences p, p'. Then

$$\frac{p}{2r} = \frac{p'}{2r'}$$

That is, the ratio of the circumference to the diameter is the same for all circles. This common ratio is denoted by π.

Proof. Suppose that the circles have the same center. (This involves no loss of generality.)



In the figure, we indicate the ith side $\overline{A_{i-1}A_i}$ of a polygon inscribed in C. To each such polygon there corresponds a polygon inscribed in C', obtained by projection outward (or perhaps inward) from the common center D. We then have

$$\triangle\, DA_{i-1}A_i \sim \triangle\, DA'_{i-1}A'_i$$

Therefore

$$\frac{A'_{i-1}A'_i}{A_{i-1}A_i} = \frac{DA'_i}{DA_i} = \frac{r'}{r}.$$

If the perimeters of our polygons are p_n and p'_n, then we have

$$p'_n = \frac{r'}{r} * p_n$$

Let

$$p = \sup P, \qquad \text{and} \qquad p' = \sup P',$$

where $P = \{p_n\}$ and $P' = \{p'_n\}$, as usual. Then

$$P' = \frac{r'}{r} * P$$

Therefore, by the preceding theorem, using $k = \frac{r'}{r}$, we have

$$\sup P' = \frac{r'}{r} \sup P, \qquad \text{or} \qquad p' = \frac{r'}{r} p, \qquad \text{or} \qquad \frac{p'}{r'} = \frac{p}{r}$$

Dividing both sides by 2, we get the equation called for in the theorem.

Mathematical Focus 2

π is an irrational number.

It was first rigorously proven that π is irrational in 1761 by Johann Heinrich Lambert. His proof involved proving if x is a nonzero rational number, then $\tan(x)$ must be irrational. Since $\tan\left(\frac{\pi}{4}\right) = 1$, it follows that $\frac{\pi}{4}$ must be irrational, thus π is irrational.

The proof that will be shown below is one that proves π^2 to be irrational, thus making π irrational. This proof is due to Ivan Niven in 1947. This proof uses the function

$$f(x) = \frac{x^n(1-x)^n}{n!}$$

Lemma. For any integer $n \geq 1$,

(i) $f(x)$ is a polynomial of the form $f(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i$ and all the coefficients c_i are integers.

(ii) For $0 < x < 1$, we have $0 < f(x) < \frac{1}{n!}$.

(iii) The derivatives $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers for all $k \geq 0$.

Proof. By expanding the binomial $(1-x)^n$ and multiplying each term by x^n, we get the polynomial

$$x^n(1-x)^n = x^n - \binom{n}{1}x^{n+1} + \binom{n}{2}x^{n+2} - \cdots + (-1)^n\binom{n}{n}x^{2n}$$

where all coefficients are integers. It then follows that (i) holds. To see that (ii) holds, observe that for any $0 < x < 1$, we have

$$0 < x^n < 1 \qquad \text{and} \qquad 0 < (1-x)^n < 1 \qquad \Rightarrow \qquad 0 < \frac{x^n(1-x)^n}{n!} < 1.$$

Let's now show that (iii) holds. From (i) it is clear that

$$f^{(k)}(0) = 0, \quad \text{if } k < n \text{ or } k > 2n.$$

For $n \le k \le 2n$, we have $f^{(k)}(0) = \frac{k!}{n!} c_k$ which is an integer. Since

$$f(x) = f(1-x),$$

we have

$$f^{(k)}(x) = (-1)^k f^{(k)}(1-x).$$

Therefore, $f^{(k)}(1) = (-1)^k f^{(k)}(0)$ is also an integer for all k.

Theorem. π^2 is irrational.
Proof. Assume that π^2 is rational, i.e., $\pi^2 = \frac{a}{b}$ for two positive integers a and b. Let

$$F(x) = b^n \left(\pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \pi^{2n-4} f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)\right). \tag{1}$$

Observe that for all $0 \le k \le n$,

$$b^n \pi^{2n-2k} = b^n (\pi^2)^{n-k} = b^n \left(\frac{a}{b}\right)^{n-k} = a^{n-k} b^k$$

is an integer. Since $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers, we see that $F(0)$ and $F(1)$ are integers. Differentiating F twice gives

$$F''(x) = b^n \left(\pi^{2n} f^{(2)}(x) - \pi^{2n-2} f^{(4)}(x) + \pi^{2n-4} f^{(6)}(x) - \cdots + (-1)^n f^{(2n+2)}(x)\right). \tag{2}$$

Observe that $f^{(2n+2)}(x) = 0$. From (1) and (2), we get

$$F''(x) + \pi^2 F(x) = b^n \pi^{2n+2} f(x) = \pi^2 a^n f(x). \tag{3}$$

By differentiation, we get

$$\begin{aligned}
\frac{d}{dx}(F'(x)\sin \pi x - \pi F(x)\cos \pi x) &= \pi F'(x)\cos \pi x + F''(x)\sin \pi x - \pi F'(x)\cos \pi x + \pi^2 F(x)\sin \pi x \\
&= \left(F''(x) + \pi^2 F(x)\right)\sin \pi x \\
&= \pi^2 a^n f(x)\sin \pi x\,, \quad \text{from (3).}
\end{aligned}$$

From the Fundamental Theorem of Calculus, we get

$$\pi^2 a^n \int_0^1 f(x)\sin \pi x\, dx = (F'(x)\sin \pi x - \pi F(x)\cos \pi x)\Big|_0^1$$

$$= F'(1)\sin \pi - \pi F(1)\cos \pi - F'(0)\sin \pi + \pi F(0)\cos \pi$$

$$= \pi\big(F(1) + F(0)\big).$$

Thus,

$$\pi a^n \int_0^1 f(x)\sin \pi x\, dx = F(1) + F(0)$$

is an integer. Since $0 < f(x) < \frac{1}{n!}$ for $0 < x < 1$, then

$$0 < f(x)\sin \pi x < \frac{1}{n!}, \qquad \text{for } 0 < x < 1.$$

Therefore, for any integer $n \geq 1$ we have

$$0 < \pi a^n \int_0^1 f(x)\sin \pi x\, dx < \frac{\pi a^n}{n!}.$$

Since for any number a, we have

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

we can choose n large enough so that $\frac{xa^n}{n!} < 1$. This gives us

$$0 < \pi a^n \int_0^1 f(x)\sin \pi x\, dx < 1$$

which is a contradiction since $\pi a^n \int_0^1 f(x)\sin \pi x\, dx$ is an integer. Therefore π^2 is irrational.

Mathematical Focus 3

π is a transcendental number and therefore is not the root of any integer polynomials.

To understand π as being a transcendental number, it may first be useful to understand what it is meant for a number to be transcendental. We will thus begin by defining what is meant for a number to be an algebraic number, which will aid us in defining a transcendental number.

Definition. If r is a root of a nonzero polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where the a_is are integers (or equivalently, rational numbers) and r satisfies no similar equation of degree $< n$, then r is said to be an algebraic number of degree n.

Algebraic Number Examples:

Every rational number is algebraic:

- 8 is algebraic because it is a solution to $p(x) = x - 8 = 0$.
- $\frac{2}{3}$ is algebraic because it is a solution to $p(x) = 3x - 2 = 0$.
- Let $\frac{a}{b}$ be any element of $\mathbb{Q}$. Then $\frac{a}{b}$ is a solution to $p(x) = bx - a = 0$.

Algebraic numbers are not necessarily rational:

- $\sqrt{2}$ is algebraic because it is a solution to $p(x) = -2x^2 + 4 = 0$.
- $\sqrt[3]{5}$ is algebraic because it is a solution to $p(x) = x^3 - 5 = 0$.
- $\frac{1+\sqrt{5}}{2}$ is algebraic because it is a solution to $p(x) = x^2 - x - 1 = 0$.
- $i = \sqrt{-1}$ is algebraic because it is a solution to $p(x) = x^2 + 1 = 0$.

It can further be noted that the sum, difference, product, and quotient of two algebraic numbers is also algebraic. The algebraic thus form a field, which is sometimes denoted by $\mathbb{A}$ or $\overline{\mathbb{Q}}$, and is the algebraic closure of $\mathbb{Q}$.

For further examples of algebraic numbers visit:

http://mathworld.wolfram.com/AlgebraicNumber.html

Let's now define what is meant for a number to be a transcendental number.

Definition. A number that is not the root of any integer polynomial is a transcendental number. These numbers are not an algebraic number of any degree.

It can thus be noted that every transcendental number must also be irrational since every rational number is algebraic. Though as was shown above, not every irrational number is transcendental.

Gottfried Wilhelm Leibniz coined the term transcendental in 1682 when he proved $\sin x$ is not an algebraic function of x but it is said that it wasn't until the 1700s when Leonhard Euler made the first conjecture of the existence of transcendental numbers. Further, it wasn't until 1844 that the existence of transcendental numbers was actually proven by Joseph Liouville. In 1851, Liousville became the first to find an example of a transcendental number. This number is known as the Liouville constant, which is provided below.

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000 \ldots$$

Liouville further showed that this number is one many characterized as a Liouville number, a category of transcendental numbers. He showed that any number that has a rapidly converging sequence of rational approximations must be transcendental.

However, all transcendental numbers are not Liouville numbers. It was proven by German mathematician Carl Louis Ferdinand von Lindemann in 1882 that the number π is a transcendental number but it wasn't until 1953 when the number π was proven not to be a Liouville number by Kurt Mahler. Mahler also proved that non-rational algebraic numbers are not easy to approximate with rational numbers.

Visit http://mathworld.wolfram.com/TranscendentalNumber.html to see a table of some well known transcendental numbers.

Lastly it should be noted that in 1873, Georg Cantor proved that transcendental numbers are non-denumerable, whereas the algebraic numbers are denumerable. This basically means that there are many more transcendental numbers than algebraic numbers, thus the Liouville numbers and π are not unique in being transcendental numbers.

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