Situation: Proving Quadrilaterals in the Coordinate Plane

Prepared at the University of Georgia EMAT 6500
Date Last Revised: 07/31/2013
Michael Ferra

Prompt

A teacher in a high school Coordinate Algebra course is teaching a unit that involves connecting algebra and geometry through coordinates. During a particular lesson in this unit students were addressing the standard that uses coordinates to prove simple geometric theorems algebraically. More specifically, the lesson was focused on investigating properties of quadrilaterals in order to identify the most specific type of quadrilateral given four vertices. In identifying the most specific type of a quadrilateral, a student questions the classifications of quadrilaterals by asking if a rectangle can be classified as a trapezoid and vice versa.

Commentary

In order to address the situation presented, the first focus addresses the definition of a quadrilateral and definitions for their specific classifications in order to differentiate how to properly distinguish among them. The remaining foci provide tools that are useful in proving the most specific naming of a convex quadrilateral given four vertices.

Mathematical Foci

Mathematical Focus 1

In Euclidean plane geometry, a quadrilateral is a polygon with four sides and four vertices.

For the purpose of being all-inclusive when discussing quadrilaterals, it should be taken into consideration that quadrilaterals could be classified by two major classifications known as complex or simple. Complex quadrilaterals are self-intersecting, i.e. two sides cross over. Simple quadrilaterals are not self-intersecting and are either convex or concave. For the basis of this focus, the discussion will majorly look at classifications of convex quadrilaterals and will define them inclusively. Let's now further examine convex quadrilaterals.

Branching off from the classification of convex, quadrilaterals can further be labeled as tangential, trapezoid and/or cyclic.

Inscriptible (or tangential) quadrilaterals are those that permit an incircle that touches all four of its sides, i.e., the four sides are tangents to an inscribed circle. A quadrilateral is tangential if and only if opposite sides have equal sums.

Trapezoids describe a quadrilateral that has at least one pair of opposite sides that are parallel.

Cyclic quadrilaterals are those that have all four vertices lying on a circumscribed circle. Further, a quadrilateral is cyclic if and only if opposite angles sum to 180°.

Further describing the more specific types of convex quadrilaterals under these three categories, certain properties overlap thus these quadrilaterals will be defined and a diagram will be provided to examine their classifications.

Kites are tangential quadrilaterals containing two pairs of adjacent sides that are congruent.
Alternate Defintion 1: Quadrilaterals in which the diagonals are perpendicular.

Parallelograms are quadrilaterals that contain two pairs of parallel sides. Based on this definition, parallelograms can be classified as trapezoids.
Alternate Definition 1: Quadrilaterals in which opposite sides are of congruent length.
Alternate Definition 2: Quadrilaterals in which opposite angles are congruent.
Alternate Definition 3: Quadrilaterals in which the diagonals bisect each other.

Isosceles trapezoids are quadrilaterals that have one pair of opposite sides that are parallel with the non-parallel sides congruent and base angles congruent. Opposite angles thus sum to 180° and since at least one pair of sides is parallel, this quadrilateral can be classified as being cyclic and as a trapezoid.
Alternate Definition 1: A trapezoid with diagonals of equal length.

A **bicentric quadrilateral** refers to a one that is both tangential and cyclic.

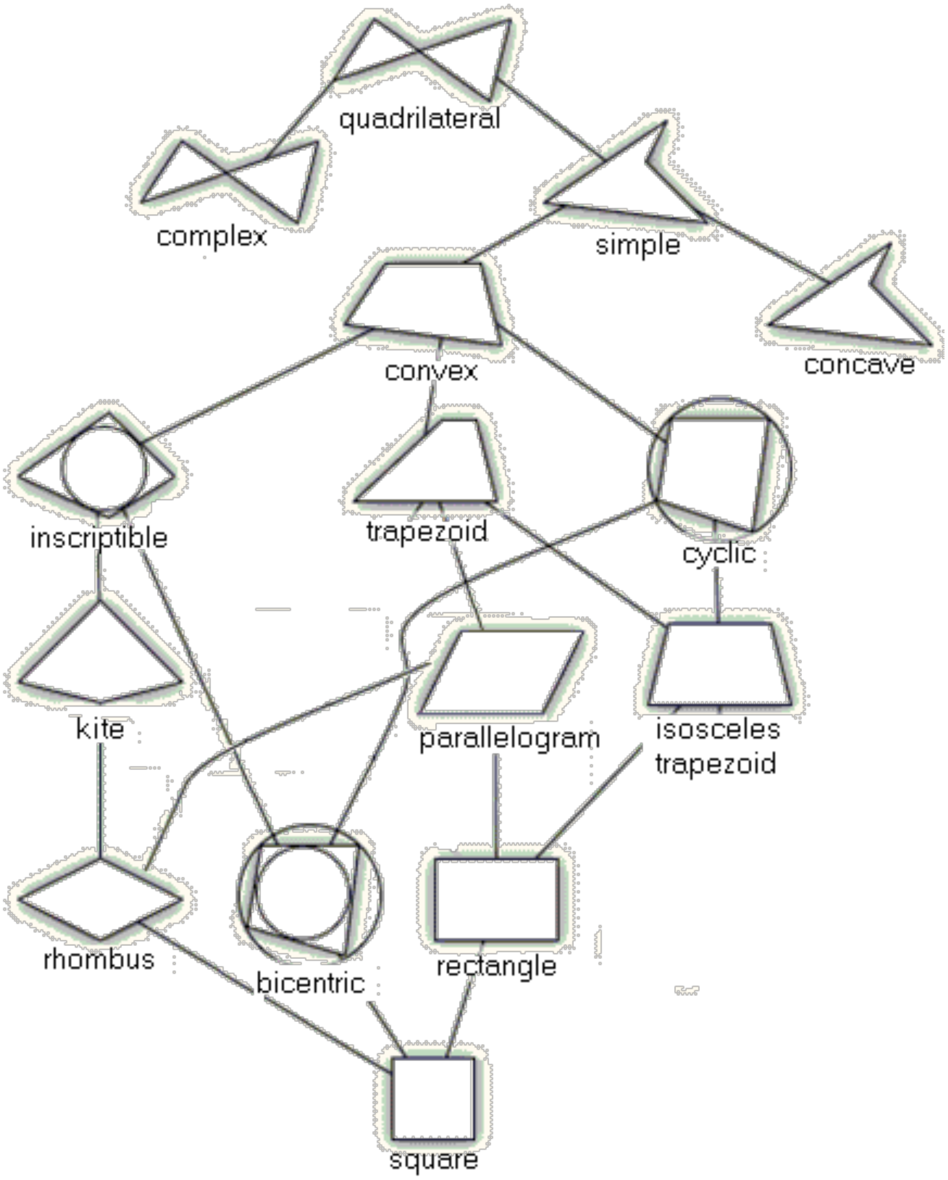
A **rhombus** is a quadrilateral with all four sides congruent. All four sides are of equal length thus two pairs of adjacent sides are congruent making a rhombus a type of kite.
Alternate Definition 1: A quadrilateral with two pairs of opposite parallel sides and opposite angles congruent. Since two pairs of sides are parallel, a rhombus is also considered a parallelogram.
Alternate Definition 2: Quadrilaterals in which the diagonals perpendicularly bisect each other.

Rectangles are quadrilaterals with opposite sides parallel and congruent.
Alternate Definition 1: A quadrilateral in which all four angles are right angles.
Alternate Definition 2: Quadrilaterals in which the diagonals bisect each other and are congruent.
Based on these definitions, rectangles are parallelograms and isosceles trapezoids.

Squares are quadrilaterals with all four sides congruent and all four angles are right angles.
Alternate Definition 1: Quadrilateral with opposite sides parallel and congruent diagonals that perpendicularly bisect each other.

Based on these definitions, a square is a rhombus and a rectangle, thus allowing the square to fit the definition of any convex quadrilateral.

For readability, here is a diagram outlining the quadrilateral classifications.



Mathematical Focus 2

In the Euclidean Plane, if $\boldsymbol{a} = (a_1, a_2)$ *and* $\boldsymbol{b} = (b_1, b_2)$ *then the distance between points* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is given by* $d(a, b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.

For the purpose of determining if a quadrilateral has congruent side lengths or congruent diagonal lengths, the distance formula can be utilized. The main focus of this prompt is examining how to determine the most specific classification of convex quadrilaterals so let's examine how the distance formula would be useful for this.

Kite: It is known that there are two pairs of adjacent sides that are congruent based on the first definition provided in focus 1, thus the distance formula would need to be applied to determine the side length of all four sides to determine congruency. Because this is the only criterion necessary for defining if the quadrilateral is a kite, this would be sufficient work. There is however an alternate definition to proving a quadrilateral is kite, thus this is not the only means in which to show a quadrilateral is a kite. This will further be addressed in focus 3.

Parallelogram: Based on Alternate Definition 1 posed in focus 1, the distance formula would be useful in showing a quadrilateral is a parallelogram by determining congruency of opposite sides. Thus the distance formula would need to be applied to determine the side length of all four sides to determine congruency. Since this definition sufficiently defines a parallelogram, no further work would need to be done. Once again, because of alternate definitions, there are other possible ways to prove a quadrilateral is a parallelogram.

Isosceles Trapezoid: Based on the first definition provided for isosceles trapezoid, it is known that the non-parallel side lengths are congruent, thus having determined the non-parallel sides, the distance formula would only be need to be applied to find those side lengths. Thus in this case, it seems we need another tool in order to determine if the figure is an isosceles trapezoid. This tool needs to help us determine which sides are parallel and which are not. However, we could use the alternate definition of an isosceles trapezoid that is defined in focus one. Since this definition states the diagonals of an isosceles trapezoid are congruent as the necessary conditions, the distance formula could be used with no other tools needed.

In the interest of brevity, the remainder of the convex quadrilaterals will have definitions that allow for the employment of using the same technique as defined above for kite, parallelogram, and isosceles trapezoid in showing a figure has adjacent or opposite sides to be congruent as well as diagonals being congruent. As with the first definition of isosceles trapezoid though, there will be some unique portion of the definition that will require us to use another tool in further defining it as that specific quadrilateral.

Mathematical Focus 3

The slope of a line is defined as the change in the y-coordinate divided by the corresponding change in the x coordinate, between two distinct points on the line.

In the Euclidean Plane, if $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ then the slope of the line connecting points **a** and **b** is given by:

$$\text{slope} = \frac{b_2 - b_1}{a_2 - a_2}$$

This formula is specifically purposeful in this prompt in determining if opposite side lengths of the quadrilaterals given are parallel, if adjacent sides are perpendicular (which results in the formation of 90° angles), and if diagonals of the quadrilateral are perpendicular. In following the definitions of the quadrilaterals presented in focus one, we can use the following statements in determining side lengths that are parallel, perpendicular or neither.

- Two nonvertical lines are parallel if and only if they have equal slopes. Vertical lines have an undefined slope, thus if two lines have an undefined slope, they are vertical and parallel. The lines described in these cases are not coincident.

- Two nonvertical lines are perpendicular if the product of their slopes is -1. If one line has a slope of 0, i.e. a horizontal line, and the other has an undefined slope, i.e. a vertical line, then these two lines are also said to be perpendicular.

Mathematical Focus 4

The midpoint of a line segment is a point on a line segment that divides it into two equal parts.

In the Euclidean Plane, if $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ then the midpoint of the line connecting points **a** and **b** is given by:

$$\text{midpoint} = \left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right)$$

This formula would be deemed beneficial in investigating a quadrilateral's definition that claims to have diagonals that bisect each other. This means that the diagonals have been divided into two equal parts. If this is the case then the intersection of the diagonals would be at the midpoint. These quadrilaterals would include any that fall under the category of a parallelogram.

References

Bogomolny, A. (n.d.). Classification of Quadrilaterals. *Interactive Mathematics Miscellany and Puzzles*. Retrieved June 27, 2013, from http://www.cut-the-knot.org/Curriculum/Geometry/Quadrilaterals.shtml