Situation 3

Parentheses vs. Brackets

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<u>Prompt</u>: Students were finding the domain and range of various functions. Most of the students were comfortable with the set builder notation, but when asked to write the domain and range in interval notation, the students were running into trouble. One particular student posed the question: "How do we know when to use a bracket and when we use a parenthesis?"

<u>Commentary:</u> Determining when to use a bracket and when to use a parenthesis can be confusing for many students. I have found even while tutoring college students that many students are still confusing the two. The mathematical foci in this situation give a brief background of domain and range and the two main ways to express it: set builder notation, and interval notation. The foci dealing with interval notation will go into more depth on the difference between brackets and parenthesis and examples to help explain it.

<u>Mathematical Foci:</u>

Mathematical Focus 1:

Domain and range are used to describe the set of values that fit a particular function.

The domain of a function is the set of input values for which the argument or function is defined. In other words, the function defines an output value for each input value of the domain. For a function whose domain is a subset of the set of real numbers and is represented on the Cartesian coordinate system, the domain is represented on the x-axis. A more formal definition is as follows: Given a function $f\colon X \rightarrow Y$, the set X is the domain of f, the set Y is the codomain of f. In the expression $f(x)$, x is the argument and $f(x)$ is the value. The argument is the input and the value is the output. The image, or range, of f is the set of all values assumed by f for all possible x; this is the set $\{f(x) \mid x \in X\}$. The image of f can be the same set as the codmain or it can be a proper subset of it. It is usually smaller than the codomain and is whole codomain if and only if f is a surjective function. A surjective function is a function in which every element y in Y has a corresponding element x in X given by $f(x)=y$. A well defined function maps every element of its domain to an element of its codmomain. The codomain is a set containing the funtion's outputs whereas the image or range is the part of the codomain which consists only of the function's outputs. For example, the function $f(x) = x^2$ maps the real numbers to the real numbers, so its codomain would be the set of real numbers, but the range is the set of positive real numbers and 0 since x^2 is always positive or 0 if x is real. So if the domain is the set of all possible values that make the function defined, the range is the set of all values that the come from inputting the domain. Thus, it makes sense to first find the domain and then determine the range. There are a few basic restrictions on the domain of functions that we need to remember. First, we cannot take the square root of a negative number. The domain of a function, say $f(x) = \sqrt{x-1}$ is found by setting whatever is under the radical equal to zero. Thus the domain of this function is all real numbers greater than or equal to 1. Intuitively, we can take the square root when x = 1 and for any number greater than x = 1 because we can take the square root of zero and all numbers greater than zero. Second, we cannot divide by zero. This means that whenever we have a function in which x is in the denominator, we must find the values of x that make the denominator equal to zero because this tells us what values of x make the function undefined. Other functions such as ln(x) and tan(x) have similar domain restrictions. Looking at the example above, $f(x) = \sqrt{x-1}$, since the domain is all real numbers greater than or equal 1, we deduce that the range is all real numbers greater than or equal to 0 since the square root of any positive number gives us a positive number or 0. Now that we have a good definition of domain and range, we can discuss better mathematical ways to express them.

Mathematical Focus 2:

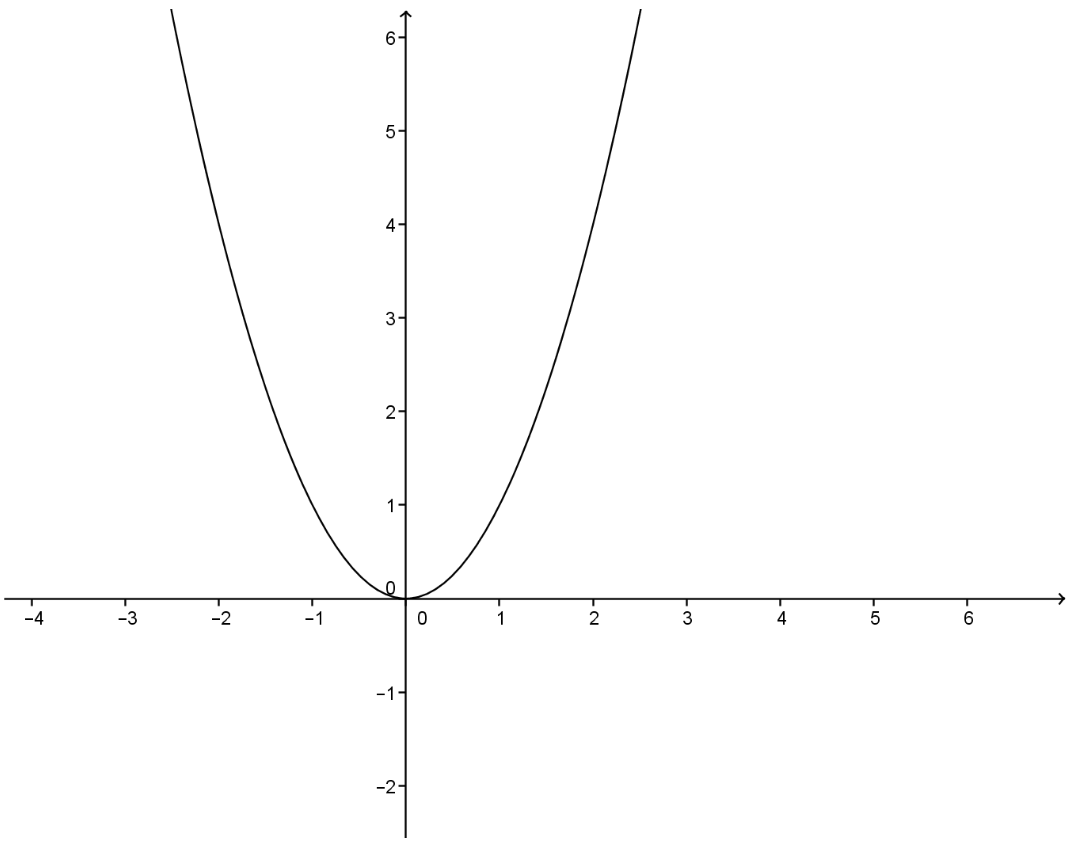
Set-builder notation and interval notation are two basic ways to express the domain and range of a function.

Set-builder notation is a mathematical notation for a set that states the properties that the particular set must have. For example, the domain of $f(x) = \sqrt{x-1}$ in set-builder notation would be: $\{x \in R | \ x \geq 1\}$. This reads, the set of all real numbers x such that x is greater than or equal to 1. Another example is $\{x \in R | \ x \neq 5\}$. This refers to all real numbers not including 5. Interval notation requires the use of parentheses and brackets. It is a type of notation that represents an interval with a pair of numbers. Parentheses and brackets are used to show whether or not a point is included or excluded. A parenthesis is used when the point or value is not included in the interval, and a bracket is used when the value is included. For example, the interval [1,6) refers to the set of all real numbers from 1 to 6 including 1 but not including 6. Interval notation is useful when representing sets of numbers, just as set-builder notation. Domain, range, intervals of increasing and decreasing etc. use intervals on the coordinate system. The intervals go from the lowest number to the highest number just as is done in the coordinate axis. The endpoint is considered an open endpoint when it is not included in the interval and a parenthesis is used. The endpoint is considered closed when it is included in the interval and a bracket is used. When both endpoints of the interval are open, the interval is called an open interval. When both endpoints of the interval are closed, the interval is considered to be closed. To combine intervals we use the union symbol: $\cup$.

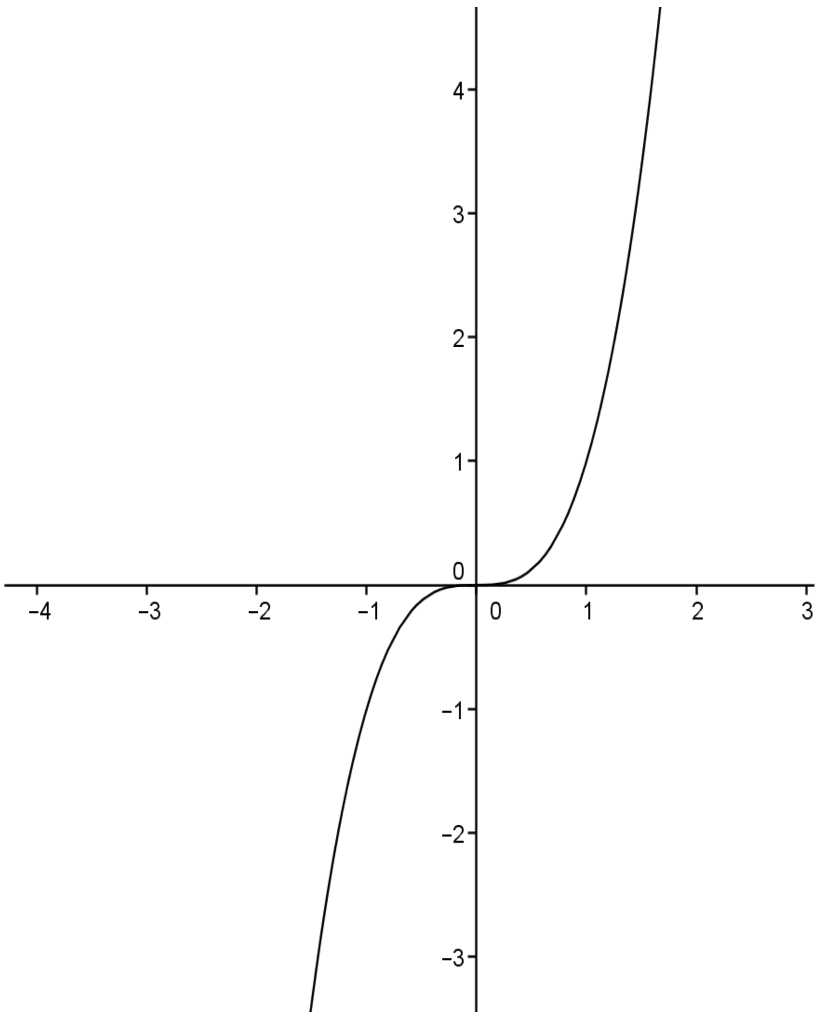
Mathematical Focus 3:

Interval notation is used to describe domain and range, to show when a function is increasing or decreasing, and to show concavity.

The domain of a function represents all the valid x values that make the function defined. The range is the set of y values that are obtained from the input values. This can be expressed in interval notation and is useful for functions defined in the coordinate plane. For example take the function $f(x) = x^2$. The domain is all real numbers, and the range is all the positive real numbers including zero. In interval notation the domain is $(-\infty, \infty)$ and the range is $[0, \infty)$. Note that the intervals are read from left to right and bottom to top in reference to the x and y axis. Also note that a parenthesis goes around the infinities and a bracket goes around 0. A bracket goes around 0 because 0 is included in the domain. This is due to the fact that we can take the square root of zero. We will get into why a parenthesis goes around infinity in the next foci. If we look at the derivative of this function, we can determine when the function is decreasing and when the function is increasing. We can also determine this from the graph of the function which is given below:



We can see by looking at the graph that the function is decreasing on the interval $(-\infty, 0)$ and increasing on the interval$(0, \infty)$. Note that now when talking about intervals of increasing and decreasing, we instead use a parenthesis around 0. This is because at the point 0, the function is neither increasing nor decreasing. The function has slope of 0 at x=0 because there is a minimum at x=0. (Intervals of increasing and decreasing are always expressed in terms of the x values.) Therefore, the point x=0 cannot be included in the interval of increasing or the interval of decreasing. Thus, a parenthesis is used. Every real number leading up to zero, from the negative and positive side, can be included, but not the point 0. Concavity is also expressed in a similar manner. Note that in this particular function, the function is concave up on the interval$(-\infty, \infty)$. If we were to look at the function $f(x) = x^3$ we get the following graph:



The function is concave down on the interval $(-\infty, 0)$ and concave up on the interval$(0, \infty)$. The point x=0 is not included because x=0 is a point of inflection and the slople is 0 at x = 0. The function is changing from concave down to concave up at 0 and thus 0 cannot be included in either interval.

Mathematical Focus 4:

A parenthesis is always used around infinity or negative infinity when describing an interval in interval notation.

Infinity and negative infinity are considered open endpoints and are therefore always expressed with a parenthesis. If we were to consider infinity to be a closed endpoint, that would mean that the value infinity would be included in the interval. Thus, for example, if we were talking about domain and range, then the interval $(-\infty, \infty)$ would mean that the function can absolutely equal negative infinity or positive infinity. However, how can a value equal infinity? Nothing is equal to infinity. Infinity and negative infinity are limiting points and are not assigned specific values. The interval does not include infinity or negative infinity because infinity cannot be contained. This is why infinity is always expressed with a parenthesis instead of a bracket.

Post Commentary: The concept of open and closed intervals and how to determine this can be confusing for students. The teacher needs to understand what interval notation is and how to relate it to the other notations that are commonly used before interval notation is taught. It is important to know the uses for interval notation in order to bring up every possible scenario. Another important fact to mention is that the use of brackets and parenthesis is conventional. Depending on the textbook or the field that one is looking at, something else might be used to express an open or closed interval. For this particular setting in a high school classroom, we use the bracket and parenthesis and examine the uses of interval notation for domain, range, increasing, decreasing, and concavity.