Situation: 180˚ in a Euclidean Triangle

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Prompt

A teacher introduces the daily lesson by listing and explaining various properties of triangles. During this introduction the teacher states that the measures of the interior angles of all triangles located on a flat plane will sum to 180 degrees. A student, startled by the concreteness of this triangular fact, quickly asked, "So there are no triangles that exist with interior measures that sum to greater than or less than 180 degrees?"

Commentary

The following set of foci use geometric and algebraic representations to prove and understand the property of triangles discussed in the prompt above. The subsequent foci will help teachers comprehend the background information needed in order to understand why the measures of interior angles of a triangle sum to 180 degrees. The interior angle measure property is fundamental to Euclidean geometry. The fifth focus was included to expand on the existence of triangles. There are cases where the sum of the interior angle measures do not add up to 180˚ implying the triangle lives in a Hyperbolic plane rather than an Euclidean plane. Furthermore, this situation is designed to further expand teachers' comprehension of triangular theorems and properties.

Side Notes:

Properties of Euclidean Triangles:

- Triangles are closed basic polygonal shapes.
- Triangles have three edges and three vertices.
- The three vertices are non-collinear meaning all three points do not live on the same line.
- The interior angle measures of a triangle always sum to 180˚.
- There are six types of triangles: scalene, right, equilateral, isosceles, obtuse, acute.
- The sum of any two side lengths will be greater than the length of the remaining side.

*For a set notation review please see the following webpage for assistance. http://www.rapidtables.com/math/symbols/Set_Symbols.htm.

*For a review of the Axioms of Plane Geometry please see the following webpage for assistance. http://www.math.brown.edu/~banchoff/Beyond3d/chapter9/section01.html.

Recall the definition of a Euclidean plane:
A plane is a quadruple (*P, L, d, m*), where:

1. *P* is a set. Elements of the set *P* are called points in *P*, and are denoted by P, A, X, etc.
2. *L* is a nonempty collection of certain subsets of *P*. Each member *l* of *L* is a subset of *P* (i.e. $l \subset P$) and *l* is called a line in *P*. Lines in *P* will be denoted by *l, k, h,* etc.
3. *d* is the distance function in *P.* This means that for every two points P, Q in *P* we are given a real number d(P, Q), called the distance from P to Q.
4. *m* is an angle measure function in *P.* This means that for every angle ∠PQR in *P* we are given a real number m(∠PQR) in the interval [0, 180], called the measure of ∠PQR.

*The quadruple (*P, L, d, m*) has to satisfy the six properties or axioms of plane geometry.

Mathematical Foci

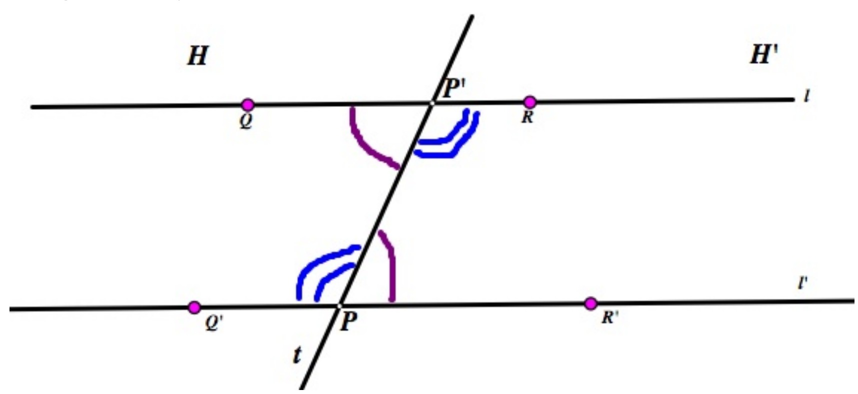
Focus 1: *If two parallel lines are cut by a transversal, then the alternate interior angles are congruent. Similarly, if two parallel lines are cut by a transversal, then the alternate exterior angles are congruent.*

Definition: Let *l* and *l′* be two distinct lines in *P* where *P* is a plane. A third line *t* is called a transversal to the lines *l* and *l′* if *t* intersects the lines *l* and *l′* at distinct points.

Definition: Let *t* be a transversal to the lines *l* and *l′*. Denote $l \cap t = \{P\}$, $l' \cap t = \{P'\}$ where *P* and *P′* are unique points.

Definition: Let *H* and *H′* be called half-planes in the Euclidean plane. Each half-plane divides the plane and is bounded by a line with infinite length.

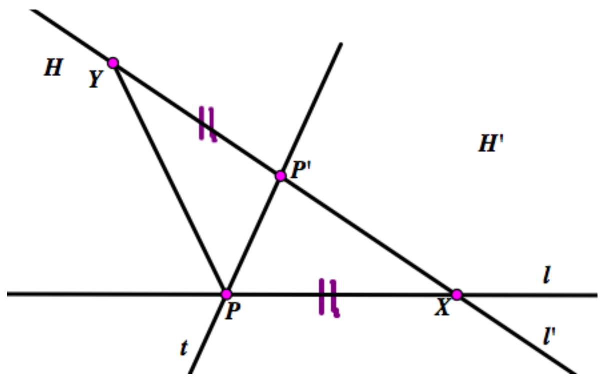
Pick any points Q and R on *l* such that Q ∈ *H, R* ∈ *H′*. Recall, *H* and *H′* are half-planes in the Euclidean plane. Pick any points Q′, R′, on *l′* such that Q′ ∈ *H, R′* ∈ *H′.* The four angles ∠R′PP′, ∠PP′R′, ∠QPP′, and ∠Q′P′P are called interior angles defined by t. There are two pairs of alternate interior angles. They are (∠RP′P, ∠Q′P′P) and (∠QP′P, ∠R′PP′).



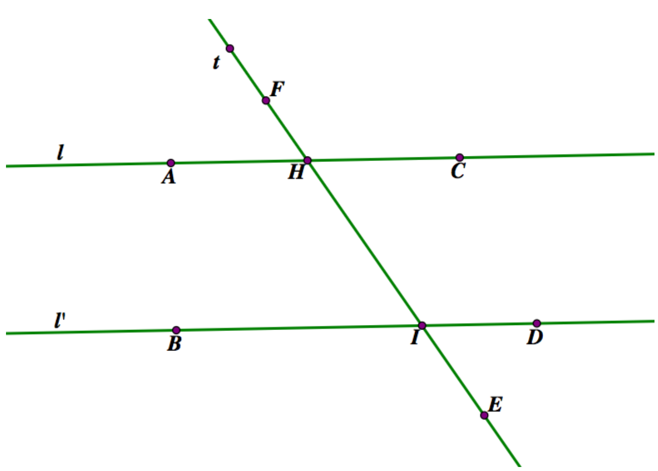
Remark for reader: If one pair of alternate interior angles consist of congruent angles, then the other pair exists as well.

The Alternate Interior Angles Theorem Proof: Let l, l′ be two distinct lines in P and let t be their transversal. Assume that alternate interior angles are congruent. Then l || l′, i.e. l is parallel to l′.

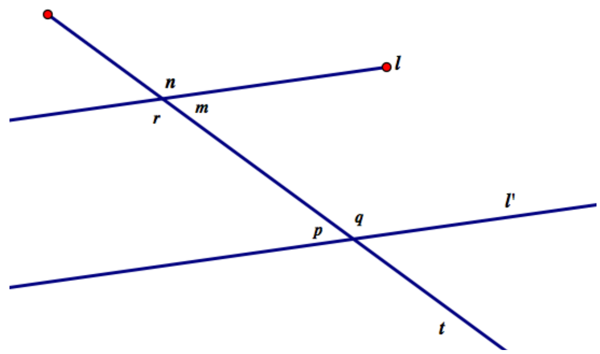
Proof 1: Assume the opposite. This means that $l \cap l' = \{x\}$. Without loss of generality, x lies in H'. Let H and H' be half-planes with boundary t. Assume $x \in H'$. We have the m(∠ YPX) = 180, so ∠YPX is straight. This implies that the point Y $\in l$ of course, Y $\in l'$. So, l' equals the lines XY which also equals l. So $l = l'$ therefore because the initial assumption is that l and l' are distinct x and y are different points because they live in two different half-planes (which have an empty intersection). In conclusion, $l \parallel l'$.



Proof 2: By the Corresponding Angle Theorem (see Focus 2 for more information) we know that m(∠CHF) and m(∠DIE) are congruent. We also know through this theorem m(∠FHA) and m(∠BIE) are congruent. We also know by the Straight Angle Theorem that the m(∠FHC) + m(∠CHI) = 180, m(∠DIE) + m(∠DIH) = 180, m(∠FHA) + m(∠AHI) = 180, and m(∠BIE) + m(∠BIH) = 180. Because we have these equalities and congruencies we can say m(∠FHA) + m(∠AHI) = m(∠DIE) + m(∠DIH) ⇒ m(∠FHA) ≃ m(∠DIH). A similar proof can be used to show that m(∠FHC) ≃ m(∠BIE). Therefore we have congruent alternate interior angles and alternate exterior angles.



Focus 2: *If two parallel lines are cut by a transversal, then the pairs of corresponding angles are congruent. Conversely, if two lines are cut by a transversal and the corresponding angles are congruent, then the lines are parallel.*



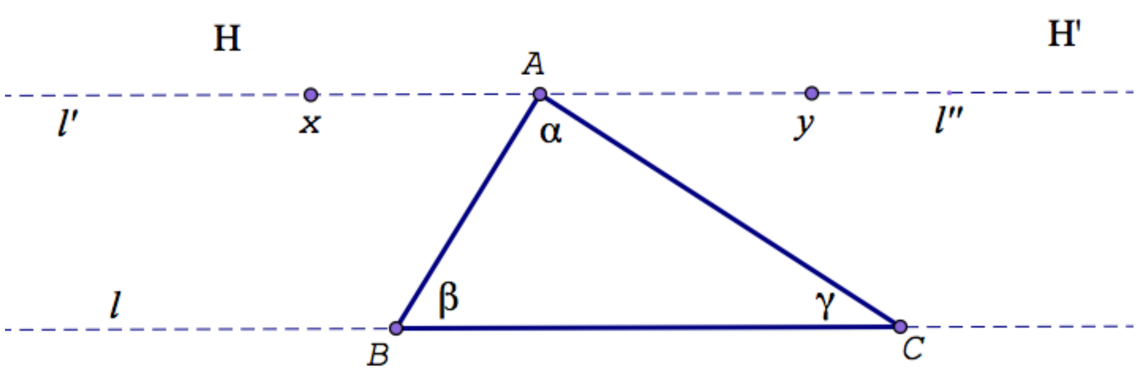
Simple proof of corresponding angles:

⇒We know through the Straight Angle Theorem that the measures of ∠n + ∠m = 180˚ and that the measures of ∠p + ∠q = 180˚. We also know through the Alternate Interior Angle Theorem (see Focus 2) that ∠m ≅ ∠ p. Because ∠n + ∠m = 180˚ and ∠p + ∠q = 180˚ we can say that the measures of ∠n + ∠m = ∠p + ∠q. Furthermore since ∠m ≅ ∠ p we can say ∠n + ∠p = ∠p + ∠q ∴ ∠n ≅ ∠q as desired. We call these angles *corresponding angles*. A similar proof can be used to show the remaining corresponding angles are congruent.

⇐ If two lines are cut by a transversal and the corresponding angles are congruent then we know by the Alternate Interior Angle Theorem that the two lines are parallel.

Focus 3: *Proofs of Angle Sum Theorem*

Let *(P, L, d, m)* be a plane. Then for every triangle in *P* the sum of the measures of all three angles in the triangle is 180˚.

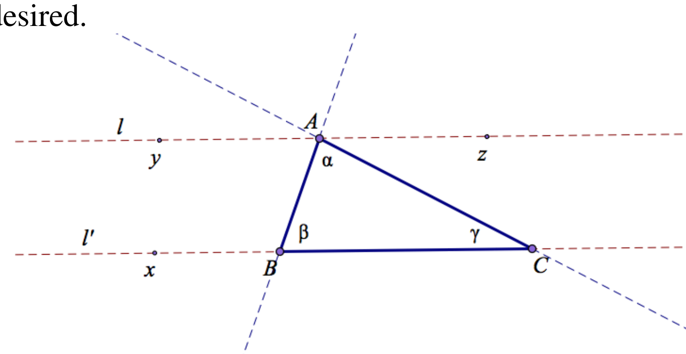


Let △ABC be any triangle. Denote α = m(∠A) and β= m(∠B) and ɣ = m(∠C). Let *H* be the half-plane with the boundary line AB such that C∉*H*. By Axiom 5 there exists a point x in H such that m(∠XAB) ≅ β. Denote l'= the line AX. By the Alternate Interior Angle Theorem we see that *l′*‖ *l* where *l* = the line BC. Let *H′* be the half-plane with boundary line AC such that β ∉ *H′*. By Axiom 5 there exists a point Y ∈ *H′* such that m(∠YAC) ≅ ɣ. Denote *l″* = the line AY. By the Alternate Interior Angle Theorem it can be concluded that l'' || *l.* Since Axiom 7E holds in our plane we find *l′*= *l″*. Therefore, α + β + ɣ = 180˚.

*Axiom 7E (Euclidean Parallel Postulate) states for each pair (*l*, P), where *l* is a line in *P* and P ∈ *P* but P ∉ *l* there exists a unique line *l′* passing through P such that *l′* || *l*. (See Focus 4 for more details on the Euclidean Parallel Postulate).

Proof of the Angle Sum Theorem in simpler words:

Given △ABC the Alternate Interior Angle Theorem can be used to prove that the measures of the interior angles of △ABC sum to 180 degrees. Observe the sketch below. A line is drawn through vertex A parallel to the line BC. The lines AB and AC are transversals to the parallel lines. By the Alternate Interior Angle Theorem we know that m(∠CBA) ≅ m(∠YAB), m(∠CAB) ≅ m(∠XBA), and m(∠BCA) ≅ m(∠ZAC). Looking at the angles ∠YAB, α, and ∠CAZ we see these angles create a straight line. Additionally, using the congruencies stated above we can use substitution and say α + β + ɣ together form a straight line, i.e. 180˚ line. Thus α + β + ɣ = 180˚ as desired.



Focus 4: *Euclid's Parallel Postulate states for each pair (l,* P*), where l is a line in P and* P ∈ *P and* P ∉ *l there exists a unique line l′ passing through* P *such that l′* || *l.*

Euclid's parallel postulate has stumped mathematicians for dozens of centuries. Efforts were made to prove the parallel postulate with Euclid's first four axioms; however, the fifth postulate still remains unproven yet not disregarded. Other attempts to prove the postulate relied on the assumption that parallel lines are separated by a constant equidistance. Euclid himself made this assumption when he wrote the fifth postulate. Although there have been centuries of failed attempts to prove Euclid's fifth postulate the world continues to reference and utilize the postulate for mathematical exploration.

Despite the comments made above there is one more point that needs to be discussed. The proposition below is a reinforcement of parallel lines in the Euclidean plane. For lines to be parallel they must either be vertical lines or nonvertical lines with the same slope. Because the proof for 180˚ triangles relies on the assumption that lines are parallel it is important to understand the two types of possible parallel lines.

Proposition: Let l and l′ be distinct lines in P. Then the following conditions are equivalent:

a. $l \mid\mid l'$
b. *either both l and l′ are nonvertical and have the same slope or both l and l′ are vertical lines.*

Proof: a ⇒ b. Assume $l \mid\mid l'$. Suppose $l = \{(x, y) \in P \mid y = mx + b\}$ and $l' = \{(x, y) \in P \mid x = a\}$ *where* $a \in \mathbb{R}$. Looking at the system

$$y = mx + b$$
$$x = a$$

Solving the system we find that the point (a, m(a) + b) lies on l and l'. This contradicts the assumption that $l \mid\mid l'$. Therefore if $l \mid\mid l'$ then either both l and l' are nonvertical and have the same slope or both l and l' are vertical lines.

Now there are two cases.

Case 1: If both lines are vertical there is nothing to prove for their x-intercepts are different thus they will never intersect.

Case 2: Prove that m = m′ in the following equations

$l = \{(x, y) \in P \mid y = mx + b\}$

$l' = \{(x, y) \in P \mid y = m'x + b'\}$

Suppose the opposite, i.e. m ≠ m′. Now look at system

$$y = mx + b$$
$$y = m'x + b'$$

Solve it and we get mx + b = m′ + b′ ⇒ x(m – m′) = b′ - b ⇒ $x = \frac{(b'-b)}{(m-m')}$. Note: We can divide by (m – m′) because we know m ≠ m′ thus (m – m′) ≠ 0. Substituting $x = \frac{(b'-b)}{(m-m')}$ into y = mx + b we get $y = m\frac{(b'-b)}{(m-m')} + b$. We have found a point that belongs to l and l', which is a contradiction with the assumption that $l \mid\mid l'$ therefore m = m′ as desired.

Proof: b ⇒ a.
Assume either l and l' are nonvertical and have the same slope or both l and l' are vertical lines.

Case 1: l and l' are vertical. Since l and l' are distinct lines a ≠ a′ and $l \neq l'$ in the lines x = a and x = a′. To prove that $l \parallel l'$ assume the opposite then there exists a point (x, y) $\in l \cap l'$. Since this point (x, y) $\in l \Rightarrow x = a$ and also since the point (x, y) $\in l' \Rightarrow$ x = a′. Thus a = a′, which is a contradiction with the initial assumption that a ≠ a′.

Case 2: Since $l \neq l'$ and b ≠ b′. Prove $l \parallel l'$.

$$l = \{(x, y) \in P \mid y = mx + b\}$$

$$l' = \{(x, y) \in P \mid y = mx + b'\}$$

Suppose $l \nparallel l'$. Then there exists a point (x′, y′) $\in l \cap l'$. Then y′ = mx′ + b and y′ = mx′ + b′.

$$\begin{array}{r} (y' = mx' + b) \\ -\ \underline{(y' = mx' + b')} \end{array}$$

Thus 0 = b – b′ so b = b′ which is a contradiction with the assumption that b ≠ b′. Therefore, If l and l' are nonvertical and have the same slope or both l and l' are vertical lines then $l \parallel l'$. This completes the proof of the proposition.

Focus 5:

If this prompt was the case where the sum of the interior angle measures did not add up for 180˚ then we would no longer be in the same geometric plane. We would no longer be in a Euclidean plane, but rather a hyperbolic plane. In hyperbolic geometry the segments of a triangle are not typically straight lines but are usually curved. Therefore, the sum of the interior angle measures is always less than 180˚. Simply stated a hyperbolic triangle is just three points connected with three different lines or curves. These lines or curves are called hyperbolic lines. Hyperbolic geometry is not entirely different from Euclidean Geometry, but the obvious differences with triangles are stated above (Bennett, n.d., p. 1).

Post Commentary

It is important for teachers to understand and know basic geometric proofs. Of course, knowing more complicated proofs is also important. However, too many times mathematics teachers are not able to recall this situation’s prompt basic geometric proof. The proof itself is not complicated, but the additional foci are necessary to broaden understanding and help trigger recall. The prompt is a straightforward question. In order to answer the prompt’s question the teacher should be aware of the possible correlated mathematics. Through the foci discussed above a teacher should be able to clarify the student’s misconceptions and confusions with the summation of the measures of interior triangular angles.

So why do we as teachers need to know this property of triangles? The bottom line is triangles are a fundamental shape in geometry. Any polygonal shape can be triangulated, i.e. the polygonal shape can be broken down into a set of triangles. Triangulation breaks down complex shapes into a set of simple shapes thus theorems involving triangles can be used to prove conjectures about complex shapes. In addition, without triangles we would not have

Trigonometry. Triangles are a staple to geometry and mathematics as a whole. Without triangles our explorations and knowledge of mathematics would be limited.

References

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