ASSIGNMENT 12 – PROBLEM 4

Generate a Fibonnaci sequence in the first column using $f(0)=1, f(0)=1,$

$f(n)=f(n-1)+f(n-2)$

a. Construct the ratio of each pair of adjacent terms in the <u>Fibonnaci sequence</u>. What happens as n increases? What about the ratio of every second term? etc.

b. Explore sequences where $f(0)$ **and** $f(1)$ **are some arbitrary integers other than 1. If** $f(0)=1$ **and** $f(1)=3$ **, then your sequence is a <u>Lucas Sequence</u>. All such sequences, however, have the same limit of the ratio of successive terms.**

PART ONE

Here is the first 20 terms of the Fibonacci Sequence in the first column of a spreadhseet:

f(n)
1
1
2
3
5
8
13
21
34
55
89
144
233
377
610
987
1597
2584
4181
6765

Question: Construct the ratio of each pair of adjacent terms in the Fibonnaci sequence.

f(n)	f(n)/f(n-1)
1	
1	1.0000000000
2	2.0000000000
3	1.5000000000
5	1.6666666667
8	1.6000000000
13	1.6250000000
21	1.6153846154
34	1.6190476190
55	1.6176470588
89	1.6181818182
144	1.6179775281
233	1.6180555556
377	1.6180257511
610	1.6180371353
987	1.6180327869
1597	1.6180344478
2584	1.6180338134
4181	1.6180340557
6765	1.6180339632

Question: What happens as n increases?

Answer: As n increases, the ratio $\frac{f(n)}{f(n-1)}$ converges to the irrational number

$$\frac{1+\sqrt{5}}{2}=1.6180339887498948482045868343656$$

Proof:

We are given that $f(n)=f(n-1)+f(n-2)$.
Now dividing each side of the equation by $f(n-1)$, we get:

$$\frac{f(n)}{f(n-1)}=\frac{f(n-1)}{f(n-1)}+\frac{f(n-2)}{f(n-1)}.$$

Simplifying we get:

$$\frac{f(n)}{f(n-1)}=1+\frac{f(n-2)}{f(n-1)}.$$

Here is the trick. Let us define the ratio $\frac{f(n)}{f(n-1)}=x$ as $n\to\infty$.
Therefore, we can say that the ratio $\frac{f(n-1)}{f(n)}=\frac{1}{x}$ as $n\to\infty$.
Moreover, the ratio $\frac{f(n-2)}{f(n-1)}=\frac{1}{x}$ as $n\to\infty$ as well.
Therefore our original equation becomes:

$x=1+\frac{1}{x}$ which can be written as $x^2=x+1$

Rewriting this equation in the $ax^2+bx+c=0$ form, we get $x^2-x-1=0.$ This is a quadratic equation. In Assignment 2, we learned that the roots of a quadratic equation by are given by using the quadratic formula by:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

In our case, $a=1, b=-1, c=-1.$ Plugging these in the equation above, we get:

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$$

Simplification yields:

$$x=\frac{1\pm\sqrt{5}}{2}$$

The negative root is not acceptable because all the terms of the fibonacci sequence are positive. Therefore the solution is

$$x=\frac{1+\sqrt{5}}{2}=1.6180339887498948482045868343656$$

This number is called golden ratio.

Now let us make sure that we understand what this means: This means that the ratio of the consecutive terms $\frac{f(n)}{f(n-1)}$ converges to $x=\frac{1+\sqrt{5}}{2}$ as $n\to\infty.$ In other words, the ratio $\frac{f(n)}{f(n-1)}$ converges to the irrational number

$1.6180339887498948482045868343656$ as $n\to\infty.$

Question: What about the ratio of every second term?

f(n)	f(n)/f(n-1)	f(n)/f(n-2)
1		
1	1.0000000000	
2	2.0000000000	2.0000000000
3	1.5000000000	3.0000000000
5	1.6666666667	2.5000000000
8	1.6000000000	2.6666666667
13	1.6250000000	2.6000000000
21	1.6153846154	2.6250000000
34	1.6190476190	2.6153846154
55	1.6176470588	2.6190476190
89	1.6181818182	2.6176470588
144	1.6179775281	2.6181818182
233	1.6180555556	2.6179775281
377	1.6180257511	2.6180555556
610	1.6180371353	2.6180257511
987	1.6180327869	2.6180371353
1597	1.6180344478	2.6180327869
2584	1.6180338134	2.6180344478
4181	1.6180340557	2.6180338134
6765	1.6180339632	2.6180340557

Answer: The ratio $\frac{f(n)}{f(n-2)}$ converges to the square of the golden ratio:

$2.61803398874989484820458683436\,56 = 1 + 1.6180339887498948482045868343656$.

$2.6180339887498948482045868343656 = 1 + x$.

$x^2 = 1 + x$

Namely as n goes to infinity, the ratio $\frac{f(n)}{f(n-2)} = 1 + \frac{f(n)}{f(n-1)}$

Question: What about the ratio of every third term?

f(n)	f(n)/f(n-1)	f(n)/f(n-2)	f(n)/f(n-3)
1			
1	1.0000000000		
2	2.0000000000	2.0000000000	
3	1.5000000000	3.0000000000	3.000000000
5	1.6666666667	2.5000000000	5.000000000
8	1.6000000000	2.6666666667	4.000000000
13	1.6250000000	2.6000000000	4.333333333
21	1.6153846154	2.6250000000	4.200000000
34	1.6190476190	2.6153846154	4.250000000
55	1.6176470588	2.6190476190	4.230769231
89	1.6181818182	2.6176470588	4.238095238
144	1.6179775281	2.6181818182	4.235294118
233	1.6180555556	2.6179775281	4.236363636
377	1.6180257511	2.6180555556	4.235955056
610	1.6180371353	2.6180257511	4.236111111
987	1.6180327869	2.6180371353	4.236051502
1597	1.6180344478	2.6180327869	4.236074271
2584	1.6180338134	2.6180344478	4.236065574
4181	1.6180340557	2.6180338134	4.236068896
6765	1.6180339632	2.6180340557	4.236067627

Answer: The ratio $\frac{f(n)}{f(n-3)}$ converges to the cube of the golden ratio, namely:

$$\frac{f(n)}{f(n-3)}=x^3 \text{ as } n\rightarrow\infty.$$

In fact, this ratio is seen to be:

$$x^3=x\cdot x^2$$
$$x^3=x\cdot(x+1)$$
$$x^3=x^2+x$$
$$x^3=(x+1)+x$$
$$x^3=2x+1=4.236067977499789696409173668 7313\ .$$

Question: What about the ratio of every forth term?

Answer: The ratio $\frac{f(n)}{f(n-4)}$ converges to the forth power of the golden ratio, namely:

$$\frac{f(n)}{f(n-4)}=x^4 \text{ as } n\rightarrow\infty.$$

In fact, this ratio is seen to be:

$$x^4 = x \cdot x^3$$
$$x^4 = x \cdot (2x+1)$$
$$x^4 = 2x^2 + x$$
$$x^4 = 2(x+1) + x$$
$$x^4 = 3x + 2$$

Question: What about the ratio of every fifth term?

Answer: The ratio $\frac{f(n)}{f(n-5)}$ converges to the fifth power of the golden ratio, namely:

$$\frac{f(n)}{f(n-5)} = x^5 \text{ as } n \to \infty.$$

In fact, this ratio is seen to be:

$$x^5 = x \cdot x^4$$
$$x^5 = x \cdot (3x+2)$$
$$x^5 = 3x^2 + 2x$$
$$x^5 = 3(x+1) + 2x$$
$$x^5 = 5x + 3$$

Question: What about the ratio of every sixth term?

Answer: The ratio $\frac{f(n)}{f(n-6)}$ converges to the fifth power of the golden ratio, namely:

$$\frac{f(n)}{f(n-6)} = x^6 \text{ as } n \to \infty.$$

In fact, this ratio is seen to be:

$$x^6 = x \cdot x^5$$
$$x^6 = x \cdot (5x+3)$$
$$x^6 = 5x^2 + 3x$$
$$x^6 = 5(x+1) + 3x$$
$$x^6 = 8x + 5$$

Question: Can you generalize this? What about the ratio of every nth term?

Answer: It converges to the nth power of the golden ratio.

$$x^n = x \cdot x^{n-1}$$
$$x^n = x \cdot [f(n-1) \cdot x + f(n-2)]$$
$$x^n = f(n-1)x^2 + f(n-2) \cdot x$$
$$x^n = f(n-1) \cdot (x+1) + f(n-2) \cdot x$$
$$x^n = [f(n-1) + f(n-2)] \cdot x + f(n-1)$$
$$x^n = f(n) \cdot x + f(n-1)$$

PART TWO

Explore sequences where $f(0)$ and $f(1)$ are some arbitrary integers other than 1. If $f(0)=1$ and $f(1)=3$, then your sequence is a Lucas Sequence. All such sequences, however, have the same limit of the ratio of successive terms.

Here is the first 20 terms of the Lucas Sequence generated in spreadsheet:

f(n)
1
3
4
7
11
18
29
47
76
123
199
322
521
843
1364
2207
3571
5778
9349
15127

Question: Construct the ratio of each pair of adjacent terms in the Lucas sequence.

f(n)	f(n)/f(n-1)
1	
3	3.000000000
4	1.333333333
7	1.750000000
11	1.571428571
18	1.636363636
29	1.611111111
47	1.620689655
76	1.617021277
123	1.618421053
199	1.617886179
322	1.618090452
521	1.618012422
843	1.618042226
1364	1.618030842
2207	1.618035191
3571	1.618033530
5778	1.618034164
9349	1.618033922
15127	1.618034014

Therefore, we see that as n increases, the ratio $\frac{f(n)}{f(n-1)}$ converges to golden ratio again.

This is expected because the recursive definition of the sequence is still valid and therefore, my proof is still valid.

Question: What about the ratio of every second term?

f(n)	f(n)/f(n-1)	f(n)/f(n-2)
1		
3	3.000000000	
4	1.333333333	4.000000000
7	1.750000000	2.333333333
11	1.571428571	2.750000000
18	1.636363636	2.571428571
29	1.611111111	2.636363636
47	1.620689655	2.611111111
76	1.617021277	2.620689655
123	1.618421053	2.617021277
199	1.617886179	2.618421053
322	1.618090452	2.617886179
521	1.618012422	2.618090452
843	1.618042226	2.618012422
1364	1.618030842	2.618042226
2207	1.618035191	2.618030842
3571	1.618033530	2.618035191
5778	1.618034164	2.618033530
9349	1.618033922	2.618034164
15127	1.618034014	2.618033922

The ratio $\frac{f(n)}{f(n-2)}$ converges to the square of the golden ratio.

My proof is valid **for any recursively defined sequence** of the form:

$f(n)=f(n-1)+f(n-2)$ with arbitray first two **nonzero** and **nonnegative** terms

$f(0)=f_0, \ f(1)=f_1$

The ratio of consecutive terms of all such sequences converges to golden ratio irrespective of how we define the first two terms $f(0)=f_0>0, \ f(1)=f_1>0$.

Therefore, Fibonnacci Sequence is a special case with $f(0)=1, \ f(1)=1$ as well as the Lucas Sequence, which is a special case with $f(0)=1, \ f(1)=3$

(See proof on pages 2-3)