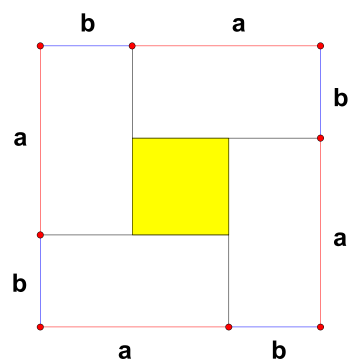
PROBLEM:

Show that for all positive values of a and b, the arithmetic mean of a and b is greater than or equal to the geometric mean, with equality if and only if a = b.

That is, show that for all a, b > 0 we have $\frac{a+b}{2} \geq \sqrt{ab}$, *with* $\frac{a+b}{2} = \sqrt{ab} \Leftrightarrow a = b$.

PROOF 1:



Assume, WLOG, that $a \geq b$. Construct a square as shown. The large square is at least as large as the four $a \times b$ rectangles (the yellow square is extra), so we have:

$$(a+b)^2 \geq 4ab$$
$$a+b \geq 2\sqrt{ab}$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$

Furthermore, the area of large square is exactly equal to the area of the four rectangles iff the yellow square's side (which is equal to $a - b$) has length zero. This occurs iff $a = b$.

Q.E.D.

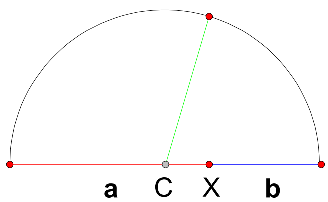
PROOF 2:

$(a-b)^2 \geq 0$	with equality iff $a = b$
$a^2 - 2ab + b^2 \geq 0$	with equality iff $a = b$
$a^2 + 2ab + b^2 \geq 4ab$	$\vdots$
$(a+b)^2 \geq 4ab$	
$a + b \geq 2\sqrt{ab}$	
$\frac{a+b}{2} \geq \sqrt{ab}$	with equality iff $a = b$

Q.E.D.

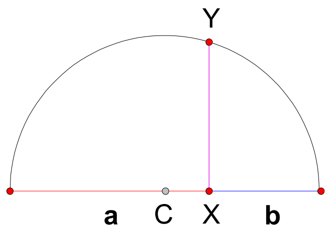
Note that this proof is extremely similar to Proof 1.

PROOF 3:



Construct a semicircle on a segment of length $a + b$.

This circle's radius is $\frac{a+b}{2}$, the arithmetic mean of a and b.



Now construct a perpendicular segment from X to the semicircle. Let Y be its point of intersection with the semicircle.

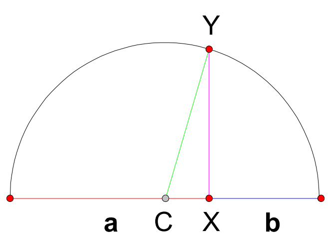
Claim: $\mathrm{XY} = \sqrt{ab}$.

Proof:

By the Pythagorean Theorem, we have:

$$\mathrm{XY}^2 = \mathrm{CY}^2 - \mathrm{CX}^2 = \left(\tfrac{a+b}{2}\right)^2 - \left(a - \tfrac{a+b}{2}\right)^2 = \left(\tfrac{a+b}{2}\right)^2 - \left(\tfrac{a-b}{2}\right)^2$$
$$= \tfrac{1}{4}\left[(a+b)^2 - (a-b)^2\right] = \tfrac{1}{4}\left[a^2 + 2ab + b^2 - \left(a^2 - 2ab + b^2\right)\right]$$
$$= \tfrac{1}{4}(4ab) = ab$$

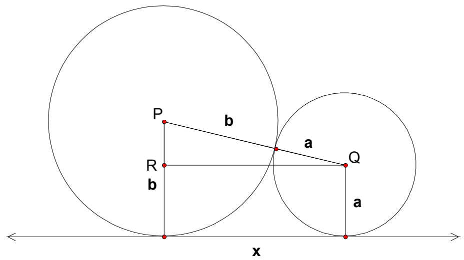
So $\mathrm{XY} = \sqrt{ab}$. The claim is true.

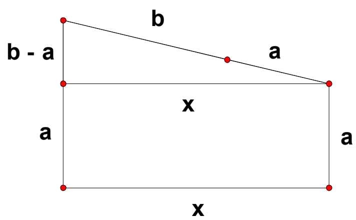


So the green segment's length is the arithmetic mean of a and b, and the purple segment's length is the geometric mean of a and b. The green segment is the hypotenuse of right triangle CXY and the purple segment is a leg, so the purple segment is always shorter than the green segment, unless the triangle is degenerate. The triangle is degenerate when X = C, which occurs when $a = b$.

Thus, the arithmetic mean of a and b is greater than or equal to the geometric mean, with equality if and only if a = b. Q.E.D.

PROOF 4:





Place two circles, one of radius *a* and the other of radius *b* tangent to a horizontal line and tangent to each other, as shown.

Let's determine the length *x*.

By the Pythagorean Theorem:

$$x^2 = (b+a)^2 - (b-a)^2$$
$$= b^2 + 2ab + a^2 - \left(b^2 - 2ab + a^2\right)$$
$$= 4ab$$

So $x = 2\sqrt{ab}$

Now we can focus on the right triangle. It has hypotenuse $a + b$ and leg $x = 2\sqrt{ab}$.
Thus, $a + b \geq 2\sqrt{ab}$, so $\frac{a+b}{2} \geq \sqrt{ab}$.

Equality holds iff the triangle is degenerate. This occurs iff the segment connecting the two centers is horizontal, which occurs iff $a = b$.

Q.E.D.