Solving Quartic Equations

Quartic equations have the general form: $ax^4 + bx^3 + cx^2 + dx + e = 0$

Quartic Equation with 4 Real Roots

Example: $3X^4 + 6X^3 - 123X^2 - 126X + 1{,}080 = 0$

Quartic equations are solved in several steps. First, we simplify the equation by dividing all terms by 'a', so the equation then becomes:

$x^4 + 2x^3 - 41x^2 - 42x + 360 = 0$, where $a = 1$ $b = 2$ $c = -41$ $d = -42$ and $e = 360$

Next we define the variable 'f': $f = c - (3b^2/8)$

Substituting the numbers from above, we get: $f = -41 - (3*2*2/8) \rightarrow f = -42.5$

Next we define 'g': $g = d + (b^3 / 8) - (b*c/2)$

Substituting the numbers: $g = -42 + (8/8) - (2 * -41 / 2) \rightarrow g = 0$

Next, we define 'h': $h = e - (3*b^4/256) + (b^2 * c/16) - (b*d/4)$

Substituting the numbers: $h = 370.56$

Next, we plug the numbers 'f', 'g' and 'h' into the following cubic equation:

$$Y^3 + (f/2)*Y^2 + ((f^2 - 4*h)/16)*Y - g^2/64 = 0$$
$$Y^3 - 21.25*Y^2 + (1{,}806.25 - 4 * 370.56)/16*Y - 0^2/64 = 0$$
$$Y^3 - 21.25*Y^2 + (1{,}806.25 - 1{,}482.25)/16*Y - 0^2/64 = 0$$
$$Y^3 - 21.25*Y^2 + 20.25*Y - 0 = 0$$

Cubic Equation Calculator
http://www.1728.com/cubic.htm

And the 3 roots of the equation are: $Y_1 = 20.25$ $Y_2 = 0$ $Y_3 = 1$

Let 'p' and 'q' be the square roots of ANY 2 *non-zero* roots (Y_1 Y_2 or Y_3).

$p=\sqrt{(20.25)} = 4.5$

$q=\sqrt{(1)} = 1$

$r= -g/(8pq) = 0$

$s= b/ (4a) = 6/(4*3) = 0.5$

Then the four roots of the quartic equation are:

$x_1= p + q + r -s = 4.5 + 1 + 0 - .5 = 5$

$x_2= p - q - r -s = 4.5 - 1 - 0 - .5 = 3$

$x_3= -p + q - r -s = -4.5 + 1 - 0 - .5 = -4$

$x_4= -p - q + r -s = -4.5 - 1 + 0 - .5 = -6$

Quartic Equation with 2 Real and 2 Complex Roots

Example: $-20x^4 + 5x^3 + 17x^2 - 29x + 87 = 0$

Simplify the equation by dividing all terms by 'a', so the equation then becomes:

$x^4 - .25x^3 - .85x^2 + 1.45x - 4.35 = 0$

Where $a = 1$ $b = -.25$ $c = -.85$ $d = 1.45$ and $e = -4.35$

$f = c - (3b^2/8)$

$f = -.873$

$g = d + (b^3 / 8) - (bc/2)$

$g = 1.342$

$h = e - (3b^4/256) + (b^2c/16) - (bd/4)$

$h = -4.263$

Next, we plug the numbers 'f', 'g' and 'h' into the following cubic equation:

$Y^3 + (f/2)*Y^2 + ((f^2 -4*h)/16)*Y -g^2/64 = 0$

$Y^3 -0.437*Y^2 + 1.113*Y -0.028 = 0$

Next, we obtain the 3 roots of this cubic equation by going to the

Cubic Equation Calculator

http://www.1728.com/cubic.htm

The 3 roots are:

$Y_1 = 0.0255$

$Y_2 = 0.206 + i * 1.0298$

$Y_3 = 0.2056 - i * 1.0299$

Let 'p' and 'q' be the square roots of ANY 2 *non-zero* roots (Y_1 Y_2 or Y_3). Whenever we have 1 Real Root and 2 complex roots, we ALWAYS choose the 2 complex roots. Find the square roots by going to the Complex Number Calculator: http://www.1728.com/compnumb.htm

$p = \sqrt{(Y2)} = 0.792 + i * 0.6498$

$q = \sqrt{(Y3)} = 0.792 - i * 0.6498$

$r = -g/(8pq) = -1.342/(8*1.0502)$

$= -0.1597$

$s = b/(4a) = 5/(4*-20) = -.0625$

Then the four roots of the quartic equation are:

$x_1 = p + q + r - s = (0.792 + i * 0.6498) + (0.792 - i * 0.6498) + (-0.1597) - (-.0625)$

$x_1 = 2*(.792) - 0.1597 + .0625$

$x_1 = 1.585 - 0.0972$

$\mathbf{x_1 = 1.488}$

$x_2 = p - q - r - s = (0.792 + i * 0.6498) - (0.792 - i * 0.6498) - (-0.1597) - (-.0625)$

$x_2 = 2*(.6498*i) + .1597 + .0625$

$\mathbf{x_2 = 0.2222 + i*1.2997}$

$x_3 = -p + q - r - s = -(0.792 + i * 0.6498) + (0.792 - i * 0.6498) - (-0.1597) - (-.0625)$

$x_3 = -2*(.6498*i) + .1597 + .0625$

$\mathbf{x_3 = 0.2222 - i*1.2997}$

$x_4 = -p - q + r - s = -(0.792 + i * 0.6498) - (0.792 - i * 0.6498) + (-0.1597) - (-.0625)$

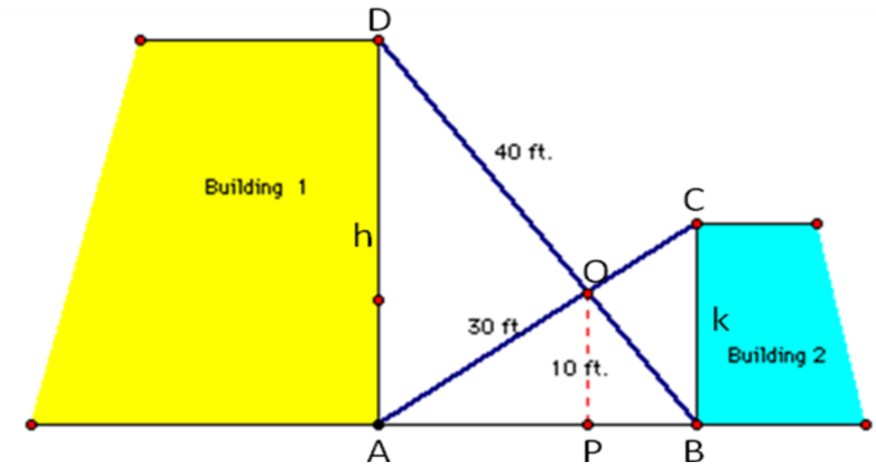
$x_4 = -2*(.792) - 0.1597 + .0625$

$x_4 = -1.585 \quad -0.0972$

$\mathbf{x_4 = -1.682}$

Source of Information: http://www.1728.com/quartic2.htm

Application of Quartic Equation Solving: How wide is the alley?



Two buildings are separated by an alley. Two ladders are placed so that the base of each ladder is against one of the buildings and reaches the top of the other building. The two ladders are 40 feet and 30 feet long. Further, they cross at a point 10 feet from the ground. How wide is the alley?

AB = Base   AP = u   PB= v

l = ladder on left wall   a = length of l = 40 ft   h = height from bottom to top of l

ll = ladder on right wall   b= length of ll = 30 ft   k = height from bottom to top of ll

c = height from bottom to intersection = 10 ft

O= intersection of the ladders

P = Intersection of height c and AB

C = Intersection of the height k and ladder ll

D = Intersection of the height h and ladder l

Since OP is parallel to DA, ΔBOP and ΔBDA are similar. $\frac{c}{h} = \frac{v}{u+v}$ Corresponding sides are proportional.

Since OP is also parallel to CB, ΔAOP and ΔACB are similar. $\frac{c}{k} = \frac{u}{u+v}$ Corresponding sides are proportional.

In ΔBDA, we can use the Pythagorean Theorem to see that $a^2 = h^2 + (u+v)^2$ (Eq. 1)

In ΔACB, we can use the Pythagorean Theorem to see that $b^2 = k^2 + (u+v)^2$ (Eq. 2)

Subtracting Eq.2 from Eq. 1, we get $a^2 - b^2 = h^2 - k^2$.

$$\frac{c}{h} + \frac{c}{k} = \frac{v}{u+v} + \frac{u}{u+v}$$

$$c\left(\frac{1}{h} + \frac{1}{k}\right) = \frac{v+u}{u+v}$$

$$c\left(\frac{1}{h} + \frac{1}{k}\right) = 1$$

$$\left(\frac{1}{h} + \frac{1}{k}\right) = \frac{1}{c}$$

$$\frac{1}{h} = \frac{1}{c} - \frac{1}{k}$$

$$\frac{1}{h} = \frac{(k-c)}{ck}$$

$$\mathbf{h = \frac{ck}{k-c}}$$

Substituting h into $a^2 - b^2 = h^2 - k^2$, we get, with some simplification:
$k^4 - 20k^3 + 700k^2 - 14000k + 70000 = 0$

Using the correct method for solving a quartic equation, we get that the real positive root is 29.379. Therefore, h = 15.16.

Now use the Pythagorean Theorem to find out how wide the alley is.

$$h^2 + AB^2 = a^2$$

$$15.16^2 + AB^2 = 40^2$$

$$AB^2 = 1370.17$$

$$\mathbf{AB \approx 37\ feet}$$