5200 fall 2009 Collineations

Here's another vector proof of concurrence of medians by vector methods.

We know that all medians concur in an equilateral triangle by symmetry. If we could find a one to one correspondence of the plane, sending the vertices of a given triangle ABC to the vertices of an equilateral triangle XYZ, and sending lines to lines, then it would follow that the intersection of two medians in ABC would correspond to the intersection of the corresponding lines in XYZ. If our correspondence also preserved scaling along each line, then it would also send midpoints of the sides of ABC to midpoints of medians of XYZ. Then it would send medians in ABC to medians in XYZ, and also send intersections of medians in ABC to intersections of medians in XYZ. Since all 3 intersections of medians in XYZ are the same point, it would follow that they must all be the same point in ABC (because the correspondence is one to one).

Now if we picture the vertices ABC and XYZ as points in three space, i.e. as vectors from the origin O to each of these points, then we know from linear algebra that these two sets of vectors {OA,OB,OC} and {OX,OY,OZ} form two different vector bases for R^3. Hence there is a linear transformation taking one basis to another, and preserving vector addition and preserving scaling. This does what we want. I.e. given two triangles ABC and XYZ in three space (with no vertex at the origin), there is a linear isomorphism of three space that takes the plane through the vertices A,B,C to the plane through the vertices X,Y,Z, and sends one triangle to the other, and sends the medians of one triangle to the medians of the other, and sending the centroid of one to the centroid of the other.