Final Project - Deriving Equations for Parabolas

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1. For a parabola with vertex at the origin and a fixed distance p from the vertex to the focus, $(0, p)$ and directrix, $y = -p$, we derive the equation of the parabolas by using the following geometric definition of a parabola:

A parabola is the locus of points equidistant from a point (focus) and line (directrix).

Let (x, y) be on the above parabola. Then, by definition, the distances to the focus and directrix - which is length of the segment connecting and perpendicular to the directrix at point $(x, -p)$ - and we get the following equation:

$$\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{(x-x)^2 + (y-(-p))^2}$$
$$\Rightarrow x^2 + y^2 - 2yp + p^2 = y^2 + 2yp + p^2$$
$$\Rightarrow x^2 = 4yp$$
$$\Rightarrow \frac{1}{4p}x^2 = y$$

2. For the parabola $y = ax^2$, we have vertex at the origin. For $x \neq 0$ and some fixed p,

$$\sqrt{(x-0)^2 + (ax^2-p)^2} = \sqrt{(x-x)^2 + (ax^2-(-p))^2}$$
$$x^2 = 4apx^2$$
$$1 = 4ap \quad \text{(since x} \neq 0\text{)}$$
$$\frac{1}{4p} = a$$

We have thus confirmed the equality of the forms $y = ax^2$ and $y = \frac{1}{4p}x^2$.
3. (Here we will actually write up #4 - more specifically, we will generalize the p-form of a parabola with vertex at (h, k) rather than the origin.)
First, given a parabola centered at the origin, we translate by k units upward to a parabola with vertex $(0, k)$.

$$y = ax^2 \rightarrow y = \frac{1}{4p}x^2 + k$$

Next, we translate in the x-direction by h units:

$$y = ax^2 + k \to y = a(x-h)^2 + k$$

This last equation is the familiar "vertex form" of a parabola. Observe that a remains the same.
4. (We here write up #3). We claim that all parabolas are similar. Suppose we have two different parabolas

$$y = a_1(x-h_1)^2 + k_1,\ y = a_2(x-h_2)^2 + k_2$$

If we translate each of these parabolas to where their respective vertices are at the origin, we will not alter the shape of the parabolas. Therefore, the first two parabolas have the same shape as their respective counterparts

$$y = a_1x^2,\ y = a_2x^2$$

By dilating the first parabola by a factor of $\frac{a_2}{a_1}$, we get

$$y = \frac{a_2}{a_1}(a_1x^2) = a_2x^2$$

which is exactly the other parabola. Similarly, the first parabola is a dilation of the second parabola by a factor of $\frac{a_1}{a_2}$. Given that we chose two arbitrary parabolas, we have now shown that all parabolas are similar.
This is counterintuitive for many. For instance, let us look at $y = x^2$ and $y = 2x^2$.

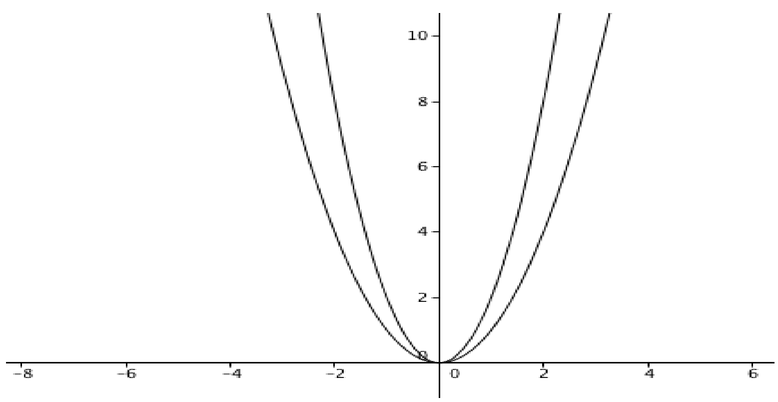


Figure 1: The graphs of $y = x^2$ and $y = 2x^2$

In Figure 1, it appears that $y = 2x^2$ creates a thinner parabola than $y = x^2$. However, if zoom out in the viewing window, we get the following graphs in Figure 2:

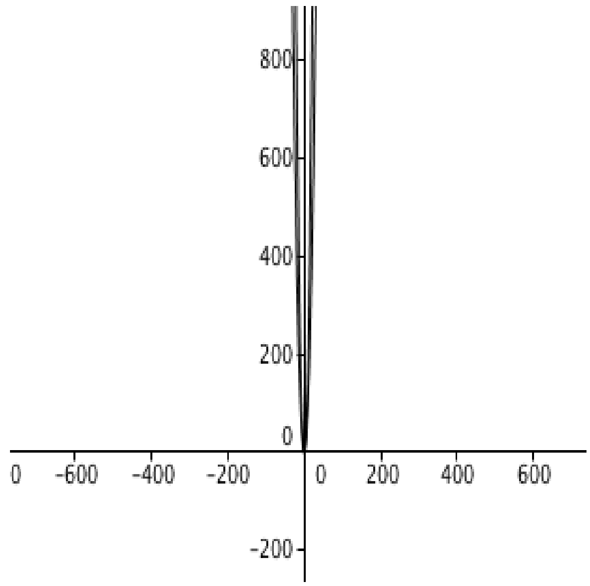


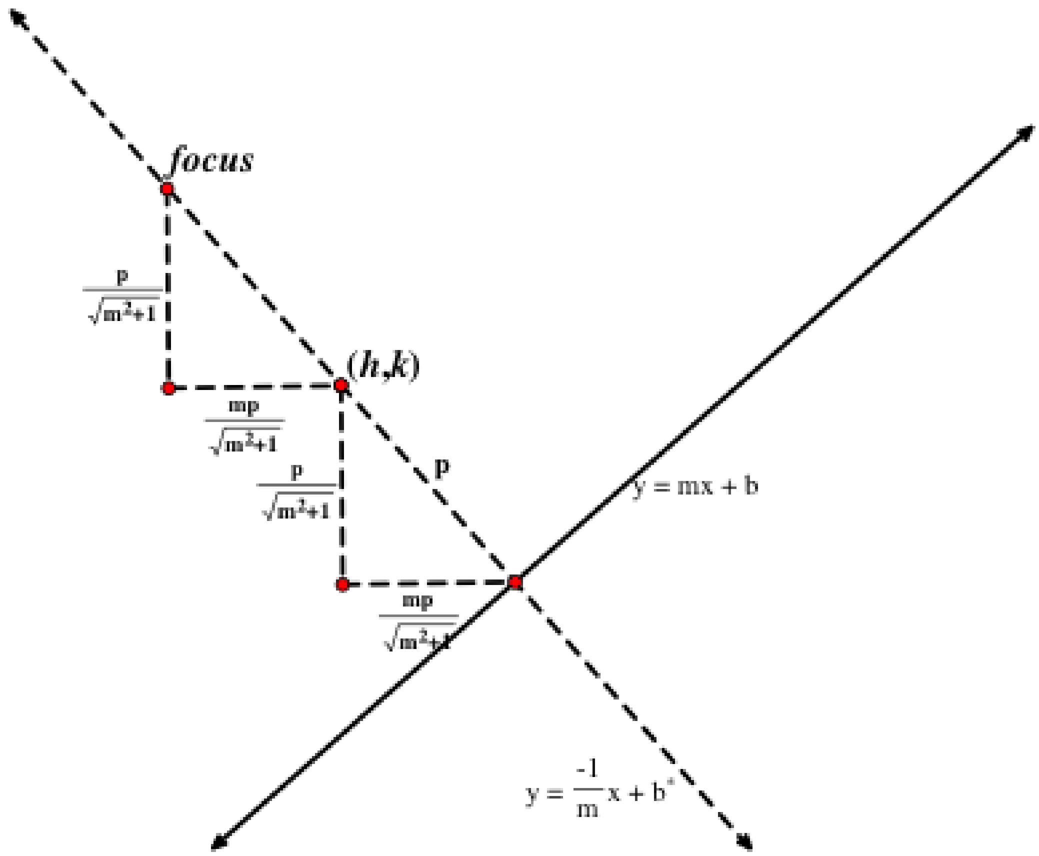
Figure 2: The same parabolas viewed in a different window.

Looking at Figure 2, the parabolas appear much more similar. Indeed, it is not so much that parabolas with varying a values have varying shapes, but that their shapes must be viewed from the proper perspective. I find this situation similar to that of circles with different radii; concentric circles do not lay flat on top of each other, but still have the same shape. Beyond graphing parabolas on the Cartesian plane, we can also consider parabolas from their geometric definition as the set of all points equidistant from a focus end directrix. Using dynamic software such as Geometer's Sketchpad or Geogebra is effective in demonstrating that the shape of a parabola (as well as the shape of other figures) is rigidly dictated *by* the geometric definition. As a point moves farther along the directrix away from the vertex, the point that is equidistant to this point and the focus necessarily must move in a parabolic shape; it is almost as if the points on the parabola are polar opposites of the magnets that are the focus and directrix.

5. If we expand our vertex form, we get

$$
\begin{aligned}
y &= a(x-h)^2 + k \\
&= a(x^2 - 2hx + h^2) + k \\
&= ax^2 - 2ahx + (ah^2 + k) \\
&= ax^2 + bx + c
\end{aligned}
$$

with $a = a$, $b = -2ah$, $c = ah^2 + k$. Note that a remains the same in general form as in vertex form.



6. We will now embark on the task of finding an equation for a parabola with directrix not parallel to either the x- or y-axis. Let (h, k) be our vertex, $y = mx + b$ our directrix with slope $m \neq 0$, and $p = p$. First, we need to find b. In order to do this, we will find a point on the directrix. We know that the point exactly p away will from the vertex will lie on the line perpendicular to the directrix with slope $\frac{-1}{m}$. Therefore, we can create the following diagrams:

Hence, one point on the directrix is $(h+\frac{mp}{\sqrt{m^2+1}}, k-\frac{p}{\sqrt{m^2+1}})$ and therefore we can calculate that

$$y = mx + b$$
$$\Rightarrow k - \frac{p}{\sqrt{m^2+1}} = m((h + \frac{mp}{\sqrt{m^2+1}}) + b$$
$$\Rightarrow b = (k - hm) - \frac{(m^2+1)p}{\sqrt{m^2+1}}$$
$$\Rightarrow b = (k - hm) - \sqrt{m^2+1}p$$

Therefore the directrix is

$$y = mx + (k - hm) - \sqrt{m^2+1}p$$

Similarly, the focus is $(h - \frac{mp}{\sqrt{m^2+1}}, k + \frac{p}{\sqrt{m^2+1}})$.

In order to derive an equation for this parabola, we need to consider a point (x_0, y_0) equidistant from the focus and directrix. We first will calculate the coordinates of the point on the directrix and the line perpendicular to the directrix through that point. We need the intersection of the line $y = \frac{-1}{m}x + (y_0 + \frac{x_0}{m})$ and the directrix:

$$\frac{-1}{m}x + (y_0 + \frac{x_0}{m}) = mx + (k - hm) - p\sqrt{m^2+1}$$

$$\Rightarrow \frac{m^2+1}{m}x = y_0 + \frac{x_0}{m} - k + hm + p\sqrt{m^2+1}$$

$$\Rightarrow x = \frac{m}{m^2+1}(y_0 + \frac{x_0}{m} - k + hm + p\sqrt{m^2+1})$$

$$\Rightarrow y = \frac{-1}{m^2+1}(y_0 + \frac{x_0}{m} - k + hm + p\sqrt{m^2+1}) + y_0 + \frac{x_0}{m}$$

We now set the squared distances from (x_0, y_0) to the focus and the directrix equal to each other:

$$(\frac{m}{m^2+1}(y_0 + \frac{x_0}{m} - k + hm + p\sqrt{m^2+1}) - x_0)^2 + (\frac{-1}{m^2+1}(y_0 + \frac{x_0}{m} - k + hm + p\sqrt{m^2+1}) + y_0 + \frac{x_0}{m} - y_0)^2$$

$$= (h - \frac{mp}{\sqrt{m^2+1}} - x_0)^2 + (k + \frac{p}{\sqrt{m^2+1}} - y_0)^2$$

$$= (\frac{mp - \sqrt{m^2+1}(x_0 - h)}{\sqrt{m^2+1}})^2 + (\frac{p - \sqrt{m^2+1}(y_0 - k)}{\sqrt{m^2+1}})^2$$

Now, this alone can serve as an equation for a parabola with directrix not at the origin. We could easily replace (x_0, y_0) with the more familiar (x, y), and all that the equation will require is p and the slope of the directrix. Further simplification does not quickly lead to any more desirable equation.

7. We can also consider the equation of a parabola centered at the origin in polar coordinates. Using the standard form and the substitution $x = r\cos\theta, y = r\sin\theta$, we have

$$y = \frac{1}{4p}x^2$$

$$\Rightarrow r\sin\theta = \frac{1}{4p}(r\cos\theta)^2$$

$$\Rightarrow 4p\sin\theta = r(\cos\theta)^2$$
$$\Rightarrow r = 4p\tan\theta\sec\theta$$

8. We can parametrize a parabola as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2pt + h \\ pt^2 + k \end{pmatrix}$$

This is justifiable if we consider the standard form of a parabola developed in (2) - $y = \frac{1}{4p}(x-h)^2 + k$. If we plug in $x = 2pt + h$, we get

$$y = \frac{1}{4p}(2pt)^2 + k$$
$$= \frac{4pt^2}{4p} + k$$
$$= pt^2 + k$$

Therefore,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2pt + h \\ pt^2 + k \end{pmatrix}$$

is a proper and simple enough parametrization of a parabola using the geometric definition. (For a parabola defined by $a, h, \& k$, the parametrization would simply be $x = t + h$ and $y = at^2 + k$.)

We have thus developed five different forms for the equation of a parabola with vertical or horizontal directrix and even one (albeit complicated) for a parabola with directrix with real non-zero slope. Each form lends itself to a different consideration of the parabola.