AM-GM Problems

Problem: Find maximization or minimization problems for which the A.M.-G.M. inequality can be used as a tool. Many of the traditional max/min problems from calculus are candidates (except that they might not be interesting . . .). Create a file of good problems.

$$\text{Arithmetic Mean} = \frac{a+b}{2}, \text{ Geometric Mean} = \sqrt{ab}$$

Part I: Use the Arithmetic Mean and Geometric Mean Inequality to show that the maximum area of a rectangular region with a given perimeter is a square.

Since the perimeter of the rectangle (i.e. a square) is given, $a + b$ must be constant.

Suppose, the perimeter $= 2c$

Then, $a + b = c$ and $a = c - b$

Arithmetic Mean$= \frac{a+b}{2} = \frac{c-b+b}{c} = \frac{c}{2}$, and

Geometric Mean$= \sqrt{(c-b)b} = \sqrt{cb - b^2}$

Now, since $Arithmatic\ Mean = \frac{a+b}{2} \geq \sqrt{ab} = Geometric\ Mean$

$$\frac{c}{2} \geq \sqrt{cb - b^2}$$

$$c \geq 2\sqrt{cb - b^2}$$

$$c^2 \geq 4(cb - b^2)$$

$$c^2 - 4cb - 4b^2 \geq 0$$

$$(c - 2b)^2 \geq 0$$

Therefore,

$$(a - b)^2 \geq 0\ldots\ \ldots\ \ldots\ \ldots \text{ since } a + b = c$$

When $a = b$ (i.e. when the rectangle is a square), the arithmetic mean is equal to the geometric mean.

Otherwise, the arithmetic mean is always greater than geometric mean.

It can be further illustrated by the GSP example here.

Part II: Use the Arithmetic Mean and Geometric Mean Inequality to show that the minimum perimeter of a rectangular with a given area is a square.

Since the area of the rectangle (i.e. a square) is given, ab must be constant.

Suppose the area $= c^2$

Then $ab = c^2$ and $a = \frac{c^2}{b}$

Arithmetic Mean $= \frac{\frac{c^2}{b}+b}{2} = \frac{c^2+b^2}{2b}$ and Geometric Mean $= \sqrt{\left(\frac{c^2}{b}\right)b} = \sqrt{c^2} = c$

$$\frac{c^2 + b^2}{2b} \geq c$$

$$c^2 + b^2 \geq 2bc$$

$$c^2 - 2bc + b^2 \geq 0$$

$$(c - b)^2 \geq 0$$

Therefore,

$$\left(\sqrt{ab} - b\right)^2 \geq 0\ldots\ \ldots\ \ldots\ \ldots \text{ since } ab = c^2$$

Hence, when $a = b$ (i.e. the rectangle is a square), the arithmetic mean is equal to the geometric mean.

Otherwise, the arithmetic mean is always greater than the geometric mean.