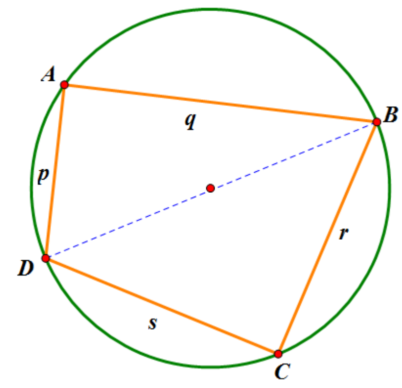
Brahmagupta's Formula

Problem: Develop a proof for Brahmagupta's Formula.



Solution: Brahmagupta's formula gives the area of a Cyclic Quadrilateral as

$$A = \sqrt{(S-p)(S-q)(S-r)(S-s)}$$

Where, $p, q, r,$ and s are the length of the sides and S is the semi-perimeter defined as, $S = \frac{p+q+r+s}{2}$.

Now, Area of the Cyclic Quadrilateral = Area of ΔADB + Area of ΔBDC

$$= \frac{1}{2} pq \sin A + \frac{1}{2} rs \sin C$$

But since $ABCD$ is a cyclic quadrilateral, $\angle DAB = 180^{\circ} - \angle DCB$

So, $\sin A = \sin C$.

Therefore, Area of Cyclic Quadrilateral = $\frac{1}{2} pq \sin A + \frac{1}{2} rs \sin A = \frac{1}{2} \sin A\,(pq + rs)$

Squaring both sides yields, $(Area)^2 = \frac{1}{4}\, sin^2 A (pq+rs)^2$

$$4(Area)^2 = (1 - cos^2 A)(pq+rs)^2$$

$$4(Area)^2 = (pq+rs)^2 - cos^2 A (pq+rs)^2 \quad \text{……………….(i)}$$

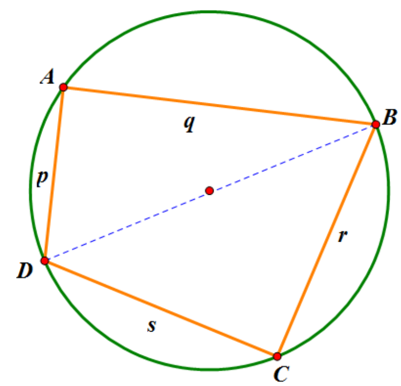
Now, let's apply law of cosine for ΔADB and ΔBDC.

For ΔADB, $BD^2 = p^2 + q^2 - 2pq \cos A$, and ……………(ii)

For ΔBDC, $BD^2 = r^2 + s^2 - 2rs \cos C$ …………………(iii)

So, combining (ii) and (iii), we have, $p^2 + q^2 - 2pq \cos A = r^2 + s^2 - 2rs \cos C$

Since, $\angle A$ and $\angle C$ are supplementary, we have $\cos C = -\cos A$



So, $2\cos A\ (pq+rs) = p^2+q^2-r^2-s^2$

$$\cos A\,(pq+rs) = \frac{1}{2}(p^2+q^2-r^2-s^2)$$

Hence, $\cos^2 A\,(pq+rs)^2 = \frac{1}{4}(p^2+q^2-r^2-s^2)^2$

Now, let's substitute this in equation (i)

$$4(Area)^2 = (pq+rs)^2 - \frac{1}{4}(p^2+q^2-r^2-s^2)^2$$

$$16(Area)^2 = 4(pq+rs)^2 - (p^2+q^2-r^2-s^2)^2$$

Using *difference of two squares* formula on the right side,

$$16(Area)^2 = (2(pq+rs)+p^2+q^2-r^2-s^2)(2(pq+rs)-p^2-q^2+r^2+s^2)$$

$$= ((p+q)^2-(r-s)^2)((r+s)^2-(p-q)^2)$$

$$= (p+q+r-s)(p+q+s-r)(p+r+s-q)(q+r+s-p)$$

Now, since $S = \frac{p+q+r+s}{2}$,

We have, $16(Area)^2 = 16(S-p)(S-q)(S-r)(S-s)$

So, $Area = \sqrt{(S-p)(S-q)(S-r)(S-s)}$.