Proof of Ceva's Theorem

Problem: Prove Ceva's theorem, that is, in any triangle ΔABC the cevians AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1 \text{ (in simple form)}$$

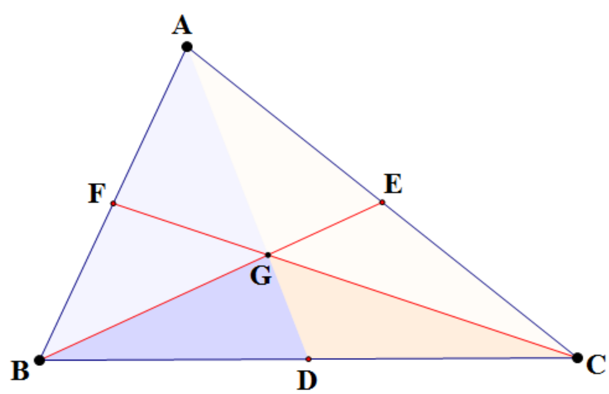
or

$$\frac{\sin BAD}{\sin DAC}\cdot\frac{\sin ABE}{\sin EBC}\cdot\frac{\sin BCF}{\sin FCA} = 1 \text{ (in trigonometric form)}$$

Note: Cevian is the line segment that connects a vertie of a triangle with the opposite side. And when three or more lines all pass through a common point, is called concurrent.

Solution: The proof of Ceva's Theorem is based on the area of triangle.

Lemma: The areas of triangles with equal altitude are proportional to the bases of the triangles.



Note: (ABC) denotes the area of ΔABC.

Let AD, BE, and CF concur at point G. So we have: $\frac{BD}{DC} = \frac{(BDA)}{(CDA)} = \frac{(BDG)}{(CDG)}$

$$\frac{BD}{DC} = \frac{(BDA)-(BDG)}{(CDA)-(CDG)} = \frac{(ABG)}{(ACG)}$$

Similarly, we can get $\frac{CE}{CA} = \frac{(BCG)}{(BAG)}$ and $\frac{AF}{FB} = \frac{(CAG)}{(CBG)}$

So, $\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA} = \frac{(CAG)}{(CBG)}\cdot\frac{(ABG)}{(ACG)}\cdot\frac{(BCG)}{(BAG)} = 1.$

For the converse, suppose $\frac{AF}{FB} = \frac{BD}{DC} = \frac{CE}{EA} = 1$ and G be the point of intersection of AD and BE. Let CG meet AB at $\bar{F}$. Then by forward argument we have

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{A\bar{F}}{\bar{F}B} = 1$$

And hence we have

$$\frac{AF}{FB} = \frac{A\bar{F}}{\bar{F}B}$$

So that both F and $\bar{F}$ divide AB in the same ratio and must therefore be the same point. Hence the theorem is proved.