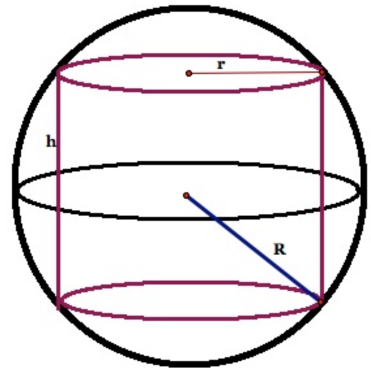
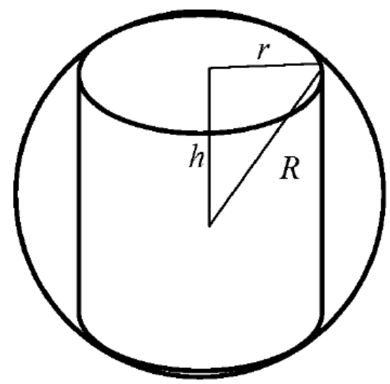
Maximum Cylinder that can be Inscribed in a Sphere

Problem: Using the AM-GM inequality, what is the maximum volume of a right circular cylinder that can be inscribed in a sphere of radius R.





Solution:

We need to maximize $V_{cylinder} = \pi r^2 h$, subject to $V_{Sphere} = \frac{4}{3}\pi R^3$

Using Pythagorean Theorem, $r^2 = R^2 - h^2$

By Arithmetic-Geometric Mean Inequality,

$$\left(V_{cylinder}\right)^2 = \left(\pi r^2(2h)\right)^2$$

$$= 4\pi^2 h^2 (R^2 - h^2)^2$$

$$= 2\pi^2 (2h^2)(R^2 - h^2)(R^2 - h^2)$$

$$\leq 2\pi^2 \left(\frac{2h^2 + (R^2 - h^2) + (R^2 - h^2)}{3}\right)^3$$

$$= 2\pi^2 \left(\frac{2R^2}{3}\right)^3$$

$$= 2\pi^2 \cdot \frac{4 \cdot 2(R^3)^2}{3 \cdot 3^2}$$

$$= \frac{1}{3} \cdot \frac{4^2 (R^3)^2}{3^2}$$

$$= \frac{1}{3}\left(\frac{4}{3}\pi R^3\right)^2$$

$$= \frac{1}{3} V_{sphere}^2$$

$$\Longrightarrow V_{cylinder} \leq \frac{1}{\sqrt{3}} V_{sphere}$$

With the equality if and only if $h = \frac{R}{\sqrt{3}}$ and $r = \frac{\sqrt{6}}{3} R$