Maximum of $f(x) = (1 - x)(1 + x)(1 + x)$

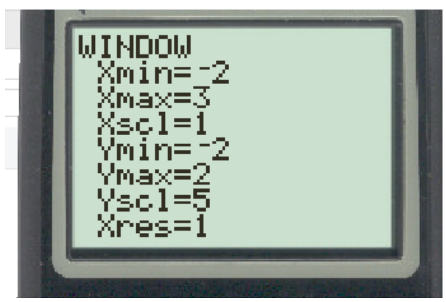
Problem: Find the maximum of $f(x) = (1 - x)(1 + x)(1 + x)$ in the interval [0, 1].

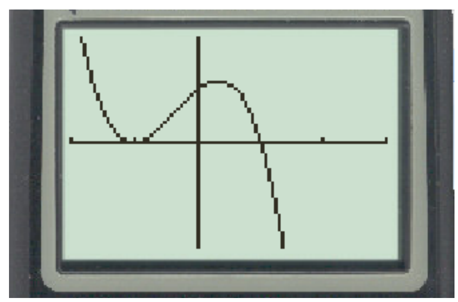
Solution:

A. Using TI-83 graphing calculator

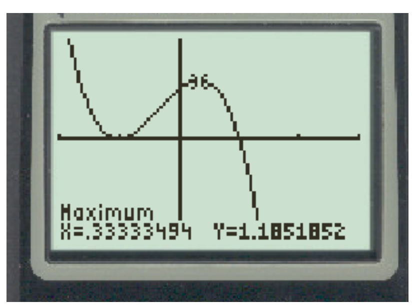
It is very easy to find the maximum of a function using TI-83 graphing calculator. First input the function on y_1 of the "Graph" option.

In order to get a better visualization of the graph of the function we need change the default window. So below is the rearrangement of the viewing window and the resulting graph.





Now pick the "maximum" command on TI-83 by following: 2nd →Trace→maximum. Then we have to pick the lower bound, upper bound, and guess/estimate of our maximum point. After the appropriate input we will get the maximum of the function as shown below:



So our maximum point is at $(0.33, 1.185)$.

B. Using AM-GM Inequality

There are three factors in this function, namely $(1-x), (1+x), and\ (1+x)$. To use AM-GM inequality we have the sum of these factors equal to a constant. So,

$$f(x) = (1-x)(1+x)(1+x)$$
$$= \frac{2(1-x)(1+x)(1+x)}{2}$$
$$= \frac{(2-2x)(1+x)(1+x)}{2}$$

By AM-GM, for non-negative terms we have:

$$\frac{1}{2}\left(\frac{2-2x+1+x+1+x}{3}\right)^3 \geq \frac{(2-2x)(1+x)(1+x)}{2}$$
$$\frac{1}{2}\left(\frac{4}{3}\right)^3 \geq f(x)$$
$$\frac{32}{27} = 1.185 \geq f(x)$$

So, 1.185 is the upper limit of the function. In other words, $f(x) \leq 1.185$ iff

$$2(1-x) = 1+x$$
$$x = \frac{1}{3}$$

C. Using Calculus

We can use calculus to find the maximum as well. For the function

$$f(x) = (1-x)(1+x)(1+x) = x^3 - x^2 + x + 1$$
$$\frac{dy}{dx} = -3x^2 - 2x + 1$$

Now, if we find the critical values: $f'(x) = 0 \Rightarrow (3x-1)(x+1) = 0$

Hence relative maximum/minimum will occur at $x = \frac{1}{3}$ or $x = -1$.

Since -1 is outside our domain, we can exclude it safely. A first or second derivative test, however, will also give us the relative maximum for the function.

Using second derivative test is somewhat easier for this problem. So, using the second derivative test,

$$f''(x) = 0 \Rightarrow -6x - 2 = 0 \Rightarrow x = \frac{1}{3}$$

So, $x = \frac{1}{3}$ is our point of inflection. Additionally, since $f''\left(\frac{1}{3}\right) < 0$, the graph concaves down at $x = \frac{1}{3}$. So we have relative maximum.

D. Using Spreadsheet

We can also use the spreadsheet to find the maximum of the function. Let's look at the value of the function on interval (0,1) with an increment of .1

A	B
x	f(x)=(1-x)(1+x)(1+x)
0	1
0.1	1.089
0.2	1.152
0.3	1.183
0.4	1.176
0.5	1.125
0.6	1.024
0.7	0.867
0.8	0.648
0.9	0.361
1	0

Now we can narrow down our initial guess. As we can see on the table, the maximum will occur between $x = .2$ and $x = .4$. Now let's start our iteration with initial value of $x = .2$ and increment of .01

A	B
x	f(x)=(1-x)(1+x)(1+x)
0.2	1.152
0.21	1.156639
0.22	1.160952
0.23	1.164933
0.24	1.168576
0.25	1.171875
0.26	1.174824
0.27	1.177417
0.28	1.179648
0.29	1.181511
0.3	1.183
0.31	1.184109
0.32	1.184832
0.33	1.185163
0.34	1.185096
0.35	1.184625
0.36	1.183744
0.37	1.182447
0.38	1.180728
0.39	1.178581
0.4	1.176

From the table, we can see that the maximum occurs at $x = .33$ and the maximum value $f(x) = 1.185$.