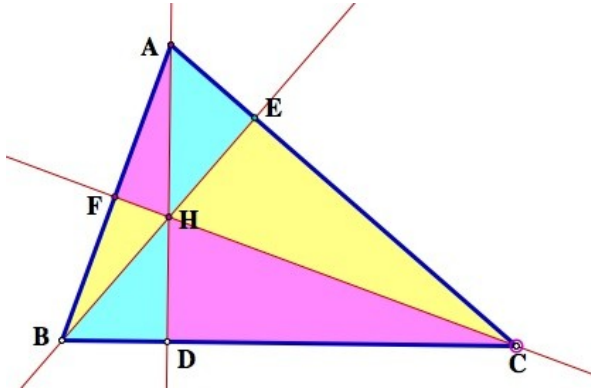


Orthocenter Problem

Problem: Given acute triangle ABC . Construct the Orthocenter H . Let points D , E , and F be the feet of the perpendiculars from A , B , and C respectively.



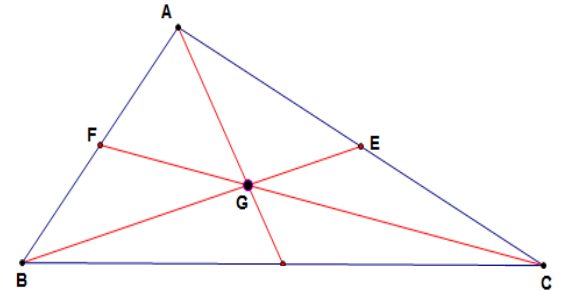
Prove:

$$1) \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} = 1 \text{ and}$$

$$2) \frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2$$

Solution: In Ceva's Theorem, we have proved that, In any triangle ΔABC , the cevians AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$



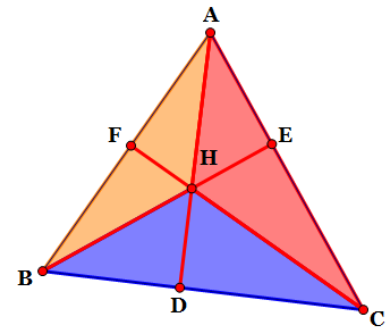
Now let's consider triangle ABC and its altitudes, $AD, BE,$ and CF . These segments appear to be concurrent at point H . Applying Ceva's Theorem to this case gives:

$AD, BE,$ and CF are concurrent if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$. Additionally, based on the similar triangles, we can conclude that

$$\Delta AFC \approx \Delta AEB \Rightarrow \frac{AF}{AE} = \frac{AC}{AB}$$

$$\Delta ABD \approx \Delta BCF \Rightarrow \frac{BD}{FB} = \frac{AB}{BC}$$

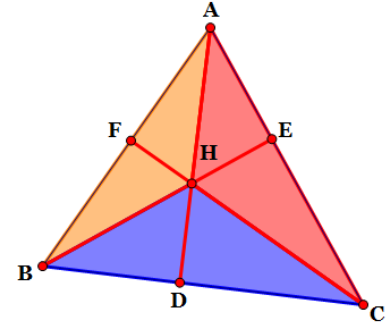
$$\Delta CAD \approx \Delta BCE \Rightarrow \frac{CE}{CD} = \frac{BC}{AC}$$



By multiplying the equations above, we get $\frac{AF}{AE} \cdot \frac{BD}{FB} \cdot \frac{CE}{CD} = \frac{AC}{AB} \cdot \frac{AB}{BC} \cdot \frac{BC}{AC} = 1$. Hence, by Ceva's Theorem, H is the point of concurrency of the altitudes of ΔABC .

First Proof:

Given $\triangle ABC$ with orthocenter H. Let points D, E, and F be the feet of the altitudes from A, B, and C respectively. We need to prove that $\frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} = 1$



Let's consider the colored triangle here. We know that the area of a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$. For each triangle, the height is the numerator of the statement: HF in $\triangle ABH$, HD in $\triangle BCH$, and HE in $\triangle CAH$. Similarly the denominator of the statement, each represent the height of the $\triangle ABC$.

So, $\frac{1}{2} \cdot BE \cdot AC$, will give us the area of $\triangle ABC$. Similar conclusions could be made for other two cases as well. This all suggests that we ought to consider the ratios of the areas of each of the small triangles to the area of $\triangle ABC$. Notice that if we add the areas of all the small triangles, it will equal the area of the large triangle. Thus our ratios should add up to one:

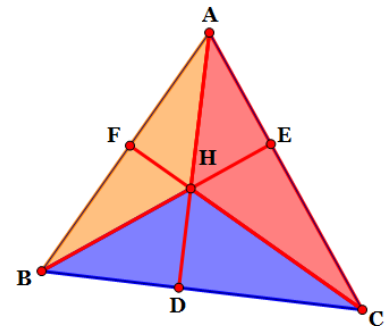
$$\begin{aligned} 1 &= \frac{\text{area}\triangle ABH + \text{area}\triangle ACH + \text{area}\triangle BCH}{\text{area}\triangle ABC} \\ &= \frac{\frac{1}{2}HF \cdot AB + \frac{1}{2}HE \cdot AC + \frac{1}{2}HD \cdot BC}{\text{area}\triangle ABC} \\ &= \frac{\frac{1}{2}HF \cdot AB}{\frac{1}{2}CF \cdot AB} + \frac{\frac{1}{2}HE \cdot AC}{\frac{1}{2}BE \cdot AC} + \frac{\frac{1}{2}HD \cdot BC}{\frac{1}{2}AD \cdot BC} \\ &= \frac{HF}{CF} + \frac{HE}{BE} + \frac{HD}{AD} \end{aligned}$$

Second Proof:

Given $\triangle ABC$ with orthocenter H. Let points D, E, and F be the feet of the altitudes from A, B, and C respectively. We need to prove that

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2$$

Now, $\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = \frac{AD-HD}{AD} + \frac{BE-HE}{BE} + \frac{CF-HF}{CF}$



$$\begin{aligned} &= \frac{AD}{AD} - \frac{HD}{AD} + \frac{BE}{BE} - \frac{HE}{BE} + \frac{CF}{CF} - \frac{HF}{CF} \\ &= 3 - \left(\frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} \right) = 3 - 1 = 2 \end{aligned}$$