Perfect Square/Cube/Fifth power

Problem: Find the smallest positive integer n such that $\frac{n}{2}$ is perfect square, $\frac{n}{3}$ is perfect cube, and $\frac{n}{5}$ is perfect fifth power.

Solution: Let n be our smallest positive integer that satisfy the condition. Now, n must be divisible by 2, 3, and 5. So, we can write

$$n = 2^a 3^b 5^c$$

To satisfy the conditions,

i) a must be odd and multiple of both 3 and 5
ii) b must be of the form $3k + 1$ and must be multiple of both 2 an 5
iii) c must be of form $5k + 1$ and must be multiple of both 2 and 3

here k is an integer.

So, $a \equiv 1(mod\ 2)$, $a \equiv 0(mod\ 3)$, $a \equiv 0(mod\ 5)$

Hence the least number possible is $a = 15$.

Then, $b \equiv 0(mod\ 2)$, $b \equiv 1(mod\ 3)$, $b \equiv 0(mod\ 5)$

Hence the least number possible is $b = 10$.

Finally, $c \equiv 0(mod\ 2)$, $c \equiv 0(mod\ 3)$, $c \equiv 1(mod\ 5)$

Hence the least number possible $c = 6$.

Combining all, we have

$$n = 2^{15} \cdot 3^{10} \cdot 5^6 = 30{,}233{,}088{,}000{,}000$$