The Proof of the Feuerbach's Theorem

Below is one of the proofs of the Feuerbach's Theorem by K. J. SANJANA. It was published in Mathematical Notes (1924), 22 : pp 11-12, Copyright Edinburgh Mathematical Society 1924.

Please note that this proof contains some statements (such as $AH = 20A_1,\ HK = 2HX,\ AI \cdot IE = 2Rr$ in line 11 of the proof), which are not obvious and can be proven separately as lemmas.

This part of the proof only proves the theorem for the inclircle. The proof for the excircles can be done in a similar manner.

Theorem: The incircle is internally tangent to the nine-point circle. The nine-point circle is tangent to all 3 excircles.

An Elementary Proof of Feuerbach's Theorem.

Let O be the centre of the circumscribing circle of $\triangle ABC$, A_1 the middle point of BC, and EA_1OF the diameter at right angles to BC. Draw AX perpendicular to BC and produce it to meet the circle in K. Let H be the orthocentre of $\triangle ABC$; join OH and bisect it in N, the centre of the nine-point circle.

Draw OY perpendicular to and bisecting AK.

Join EA, which bisects $\angle BAC$ and contains the incentre I; draw ID, NM perpendicular to BC. Join AF and draw AG perpendicular to EF; also draw PIQ parallel to BC and meeting EF in P and AX in Q.

Then we have $AH = 2OA_1$, $HK = 2HX$, $AI \, . \, IE = 2Rr$.

Also from similar triangles $\dfrac{PI}{IE} = \dfrac{FG}{AF}$ and $\dfrac{IQ}{AI} = \dfrac{AF}{FE}$.

Thus $\dfrac{PI \, . \, IQ}{AI \, . \, IE} = \dfrac{FG}{FE}$, so that $\dfrac{PI \, . \, IQ}{2R \, . \, r} = \dfrac{FG}{2R}$, and $PI \, . \, IQ = r \, . \, FG$.

Now the projection of IN on $FE = ID - NM = r - \frac{1}{2}(OA_1 + HX)$

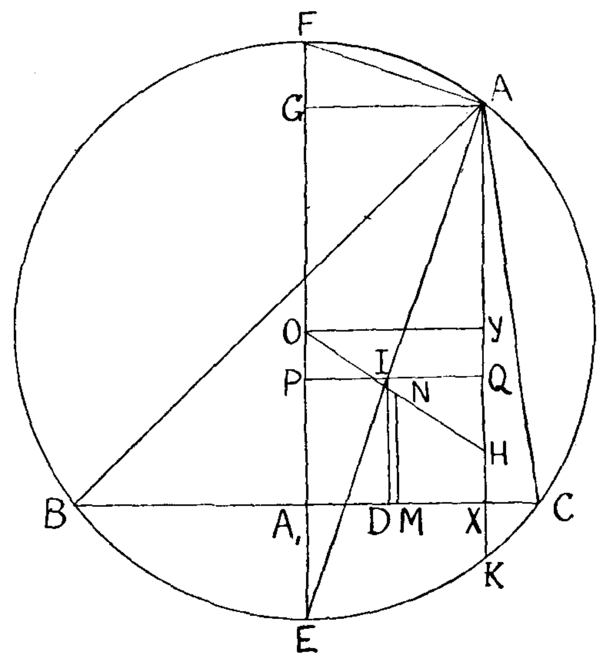
$$= r - \tfrac{1}{4}(AH + HK) = r - \tfrac{1}{2}AY.$$

Hence the square of this projection $= r^2 - r \cdot AY + \frac{1}{4}AY^2$

$$= r^2 - r \cdot GO + \tfrac{1}{4}AY^2 \quad \ldots\ldots\ldots (1)$$

Again, the square of the projection of

$$IN \text{ on } BC = DM^2 = A_1M^2 - A_1D \cdot DX$$
$$= \tfrac{1}{4}A_1X^2 - PI \cdot IQ$$
$$= \tfrac{1}{4}OY^2 - r \cdot FG \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$



Adding the results (1) and (2) we get

$$I_1N^2 = \tfrac{1}{4}(AY^2 + OY^2) - r(FG + GO) + r^2$$
$$= \tfrac{1}{4}R^2 - r \cdot R + r^2.$$

Thus $IN = \frac{1}{2}R - r$, and the theorem is proved for the incircle. The proof for an excircle proceeds on exactly similar lines.

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