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ORIGINAL ARTICLE

Building on mathematical events in the classroom

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Abstract Mathematical events from classrooms were used as stimuli to encourage mathematical discussion in two groups of mathematics teachers at the secondary level. Each event was accompanied by an analysis of mathematics that would be useful to the teacher in such a situation. The Situations, mathematical events and analyses, were used originally to create a framework describing the Mathematical Proficiency for Teaching at the Secondary Level, and then they were used with both Prospective and Practicing teachers to validate the framework. Teachers involved in the validation research claimed that the process was instructional. The process is explained, and teachers' quotes provide evidence that the experience provoked changes in teachers' understanding of mathematics. This process, which builds on mathematical events from the classroom, holds potential as a professional development experience that helps teachers expand their expertise in teaching mathematics.

Keywords Teacher learning · Cases · Secondary mathematics · Prospective teachers · Professional development

1 Introduction

Mathematical event: In a geometry class, after a discussion about circumscribing circles about triangles, a student asked, "Can you circumscribe a circle about any polygon?"

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In mathematics classrooms at the secondary level, fascinating mathematical events occur almost every day. An event might be a student question or comment; it might be an ambiguous statement in a textbook or a teacher's comment. Often, these events provide unusual opportunities to explore an idea, connect mathematical ideas, or reinforce ideas that are building within the minds of mathematical learners. Teachers need to have the expertise to take advantage of these potentially powerful opportunities. Taking advantage of mathematical events requires pedagogical knowledge and pedagogical content knowledge (Shulman 1986), but the key component of this expertise is mathematical proficiency for teaching (Wilson and Heid 2010). In this study, we have focused on Mathematical Proficiency for Teaching at the Secondary Level (MPTS).

Teachers need to make important mathematical decisions as to whether or not to alter their planned lessons to pursue unexpected opportunities. To make good decisions, teachers need a particular kind of expertise which includes a deep mathematical knowledge that allows them to recognize the opportunity, weigh its merits, and skillfully pursue or dismiss the opportunity. We argue that teachers require a mathematical proficiency that is specialized to enhance the mathematical learning of secondary students. We call this MPTS. In this paper, we describe MPTS, explain its research foundations, connect it to teacher expertise, and report perspectives of Practicing and Prospective teachers as they discuss mathematical events in focus groups. Our observations of teachers, as well as their comments, led us to believe that engaging in such discussions had the potential to identify and develop teachers' expertise related to MPTS. We argue that engaging in group discussions of mathematical events from the classroom may provide a meaningful professional development experience that develops teachers' mathematical expertise.

2 Background

There are many interpretations and characterizations of expertise in teaching. One might focus on the teacher's classroom practice or pedagogical moves. One might focus on the level of education a teacher has attained and the specialized coursework it included. We chose to study expertise as it relates to a teacher's mathematical knowledge and how it is used in practice. Most other descriptions of such expertise focus on the mathematical knowledge of elementary teachers. Believing that teaching secondary school mathematics may require different knowledge and skills than teaching elementary mathematics, we created a framework to describe the proficiency useful for teaching mathematics at the secondary level.

2.1 Previous characterizations of knowledge and expertise

Teacher knowledge has been acknowledged as influential in teaching and student learning (Fennema and Franke 1992). Shulman (1986) emphasized the role of knowledge of specific content in the profession of teaching because previous research on teaching had attended more to other elements of teaching practice. Research revealed that teachers' knowledge of subject matter affects the instructional process—what and how they teach (e.g., Ball and Feiman-Nemser 1988; Leinhardt and Smith 1985; Thompson 1984). Numerous studies and documents (e.g., Ball, Thames and Phelps 2008; Brown and Borko 1992; Leinhardt and Smith 1985; Ma 1999) agree that knowledge of subject matter is an essential component of teacher knowledge in teaching and that teachers need to develop a solid understanding of mathematics to teach mathematics well.

Shulman (1986) developed a theoretical framework for teachers' content knowledge, organizing it into three dimensions: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. By subject matter knowledge, Shulman (1986) meant knowledge of the facts or concepts of a discipline and understanding of the substantive and syntactic structures of the subject matter. Ma's (1999) notion of profound understanding of fundamental mathematics and Fennema and Franke's (1992) notion of knowledge of content of mathematics relate to Shulman's description of subject matter knowledge. Ball et al. (2008) differentiated mathematical knowledge for teaching from the knowledge needed by other mathematical professions. They proposed a refinement to Shulman's model, subdividing subject matter knowledge into three domains: common content knowledge, specialized content knowledge, and horizon content knowledge (p. 403). Common content knowledge is described as "the mathematical knowledge known in common with others who know and use mathematics" (p. 403), which closely relates to Shuman's original notion of subject matter knowledge (Hill, Ball and Schilling 2008). However, they acknowledged the difficulty in discerning between common and specialized knowledge in specific situations in their model.

The knowledge that is useful to mathematics teachers encompasses disciplinary content knowledge, but it must also include a special kind of knowledge (Shulman 1986; Ball et al. 2008). The authors of *Adding It Up* described five strands of mathematical proficiency, especially for teachers of young children: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (National Research Council (NRC) 2001, p. 116). Even (1990) developed a general framework for such subject matter knowledge at the secondary level by identifying seven critical aspects of a mathematical topic: essential features, different representations, alternative ways of approaching, the strength of the concept, basic repertoire, knowledge and understanding of the concept, and knowledge about mathematics. The seven dimensions in Even's framework relate to the five components of mathematical proficiency for teachers that were proposed by the NRC.

Research studies indicate that often teachers do not have the comprehensive and well-articulated mathematical knowledge that is needed to teach mathematics (Brown and Borko 1992; Even 1990; Stein et al. 1990), but it is not clear exactly what this special knowledge is and how it can be learned. Numerous researchers (e.g., Ball et al. 2008; Brown and Borko 1992; Even 1990; Ferrini–Mundy and Findell 2001; National Council of Teachers of Mathematics 1991; Zaslavsky and Leikin 2004) argued that the collegiate mathematics required for teacher candidates and K-12 school mathematics were disconnected. There is agreement that teachers need profound mathematical knowledge, but it is less clear exactly what that knowledge includes and how teachers might begin to develop it. In an effort to characterize that mathematical knowledge at the secondary level, we and other researchers developed a framework for MPTS. The following sections describe the framework, research validating the framework, and a model for professional development arising from the research.

2.2 Framework for MPTS

The framework for MPTS (Wilson and Heid 2010) describes the nature of a specialized mathematical knowledge and the necessary proficiency for using this knowledge. In developing this framework, we and others selected mathematical events that occurred in secondary schools or in mathematics teacher preparation classes. Using these events, mathematicians and mathematics educators described the mathematical knowledge that would help teachers recognize an interesting event, understand multiple points of view related to the event, and make decisions about what mathematical ideas to pursue. For example, in the mathematical event about circumscribing polygons that is described at the beginning of this paper, researchers agree that teachers will find it useful to know that (1) every triangle is cyclic and this insight is a core idea, (2) a convex quadrilateral in a plane is cyclic if and only if its opposite angles are supplementary, (3) concave quadrilaterals are never cyclic, and (4) every planar regular polygon is cyclic, but not every cyclic polygon is regular. Although there are more mathematical ideas that are related to the student's question, knowledge around these four ideas is very useful for leading an investigation of circumscribing polygons. The mathematics educators organized the event and accompanying mathematical ideas into a tool that they called a "Situation."

Each Situation consisted of three parts: *Prompt*, *Commentary*, and *Mathematical Foci*. The *Prompt* is a brief description of a mathematical event and includes students' or teachers' questions and insights. The *Foci* describe the various aspects of mathematical proficiency that a group of mathematicians and mathematics educators found to be relevant to someone experiencing the mathematical event described in the prompt. Each focus contains an italicized statement that clarifies the major mathematical concepts associated with the Prompt. The *Commentary* discusses the rationale for the Prompt and a summary of key ideas underlying the Foci. An abbreviated version of a Situation is given in Fig. 1. This abbreviated version contains the prompt and commentary in their entirety, the first mathematical focus, and only an italicized summary statement from the second and third foci. In the original version, these foci are elaborated.

By analyzing more than 50 of these Situations, the group of mathematics educators built a framework that described and organized mathematical proficiencies that are useful to mathematics teachers at the secondary level. The framework is divided into three sections that are not independent; rather, they are different dimensions of mathematical proficiency. Figure 2 provides subcategories for each of the three dimensions—mathematical ability, mathematical activity, and the mathematical work of teaching. The first category includes mathematical knowledge and the ability to use mathematical knowledge; the second category includes processes for doing mathematics. Proficiencies in the third category are particularly useful in the teaching of mathematics and distinguish the use of mathematics for teaching from other uses of mathematics. Mathematics teachers not only need to know mathematics and mathematical processes, but they also need to know mathematics in a way that is useful for helping someone else become proficient in mathematics.

The ideas contained in the framework arose from an analysis of the Situations, but they have also drawn on ideas that have been useful in describing mathematical knowledge at the elementary level, such as strategic competence and adaptive reasoning from *Adding It Up* (NRC 2001). The ideas in the framework reflect the spirit of work by Even (1990) at the secondary level and incorporate multiple aspects of mathematical knowledge. In the framework, we attempted to capture the unique nature of MPTS as specified by Shulman (1986) and exemplified by Ball et al. (2008) and Hill et al. (2008) at the elementary level.

2.3 Teacher expertise as characterized by the MPTS framework

By design, the MPTS framework describes mathematical knowledge that is useful for teaching mathematics at the secondary level. Teacher preparation programs require mathematics courses at a level of mathematics that exceeds what teachers will teach at the secondary level, and certification programs in the USA require specific advanced mathematics courses such as abstract algebra and advanced calculus. A growing number of school districts expect high school teachers to have a baccalaureate degree in mathematics and encourage them to continue learning mathematics in graduate programs. Mathematics instructors tend to focus on the mathematical proficiency that is characterized in the first category of the framework, mathematical ability. Prospective teachers learn skills and concepts; they become strategically competent by reasoning and persevering, and they occasionally learn historical or cultural knowledge.

Monk (1994) found that student achievement was not correlated with the number of mathematics courses that their teacher had taken. A possible explanation for this unusual finding may be that although mathematics courses require expertise in mathematics ability (Category 1), they often do not place an emphasis on Category 2 of the framework, mathematical activity. Much of the mathematical activity is demonstrated by the instructor and often followed by the Prospective teachers in the course without an analysis of the process, without noticing implicit but important features, and without an opportunity to create their own approaches. Certainly, instructors expect teachers to conjecture and to prove or disprove their conjectures, but they rarely require them to actually study, discuss, or

Fig. 1 Example of a Situation (Exponents Situation)

Situation 21: Exponential rules

Prompt

In an Algebra II class, students had just finished reviewing the rules for exponents. The teacher wrote $x^m \bullet x^n = x^5$ on the board and asked the students to make a list of values for m and n that made the statement true. After a few minutes, one student asked, "Can we write them all down? I keep thinking of more."

Commentary

Since $x^m \bullet x^n = x^{m+n}$, making a list of values that make the statement true is equivalent to finding all m and n with $m + n = 5$ (i.e., $n = 5 - m$) for any sets of values for which x^m and x^{5-m} are defined. The relevant mathematics in this Situation reaches beyond the basic rules for exponents into issues of the domains of the variables in those rules. The exponent rule $x^m \bullet x^n = x^{m+n}$ is applicable and is key to deciding how many solutions there will be. However, applying this rule beyond the usual context of positive bases and positive exponents to that of other number systems (such as the set of integers or rational numbers) requires consideration of the domains of the base and the exponents. In the following Foci, symbolic, numeric, and graphical representations are used to highlight that there are particular values in the domains of both x and m for which x^m is not a real number. In Foci 2 and 3, the domains of x and m are extended beyond the positive numbers and x^m is examined in these new domains.

Mathematical Foci

Focus 1: Defining the domains of x and m clarifies the rule $x^m \cdot x^n = x^{m+n}$ as applying to expressions with positive bases and real exponents.

The exponent rule $x^m \cdot x^n = x^{m+n}$ may be applied to the equation $x^m \cdot x^n = x^5$ if the value of x is restricted to positive numbers. That is, if $x > 0$, and m and n are real numbers, then $x^m \cdot x^n = x^{m+n}$.

If the values of m and n are restricted to natural numbers (counting numbers greater than 0), then there is a finite number of solutions for (m,n): (1,4), (2,3), (3,2), and (4,1). If m and n are not restricted to the natural numbers, then thereare infinitely many solutions for (m,n), since there are infinitely many solutions to the equation $m + n = 5$. Thus, m and n could be any two integers whose sum is 5.

The restriction $x > 0$ is necessary if the value of each term in the equation is restricted to the set of real numbers. If, for example, $m = 1.5$, then, $x^m = x^{1.5} = x^{3/2} = \sqrt{x^3} = |x|\sqrt{x}$. For this result to be a real number, x cannot be negative. Some restriction also may be necessary when x is zero, for example, if $m=-1$ and $x=0$, then x^m is undefined.

Focus 2: When the domain of x is extended to the negative real numbers or 0, then the domain of m is limited to those values for which x^m is defined. A more general principle is that extending the domain of a function of two variables might mean restricting the domain of one of the variables.

Focus 3: If the domain of m is extended, then the domain of x is limited to those values for which x^m is defined.

critique the act of proving. Teachers gain experience graphing functions and symbolically representing complex ideas such as the derivative of a function, but they usually do not spend much time assessing the representation or noticing the advantages of one representation over another.

Proficiency in mathematical activity is a critical part of teacher expertise at the secondary level, because teachers are enculturating students into the practice of mathematics. They need to be prepared to respond to students who want to understand connections between graphical representations and symbolic representations. Teachers need to be able to assess student-created definitions or procedures, knowing when they are sufficient and when they break down. Teachers need expertise to recognize a valid but redundant proof as students begin to develop the sense of an elegant proof.

Category 3 of the framework, the mathematical work of teaching, is rarely addressed in preparation of Prospective teachers or in professional development for Practicing teachers, but it is critical to developing their expertise. The mathematical work of teaching is about mathematics, but it is a specialized mathematics that is not addressed in typical mathematics courses that are charged with serving other professionals, in addition to mathematics teachers. The mathematical work of teaching may not be taught in courses focusing on pedagogy, because it is about mathematics rather than the pedagogy associated with teaching mathematics. Expert teachers have the mathematical knowledge to probe someone else's mathematical thinking. They have the mathematical knowledge to ask the right question, provide the optimal example, provide counterexamples, or find

1. **Mathematical ability**
   - Conceptual understanding
   - Procedural fluency
   - Strategic competence
   - Adaptive reasoning
   - Productive disposition
   - Historical and cultural knowledge
2. **Mathematical activity**
   - Mathematical noticing
     - Structure of mathematical systems
     - Symbolic form
     - Form of an argument
     - Connect within and outside mathematics
   - Mathematical reasoning
     - Justifying/proving
     - Reasoning when conjecturing and generalizing
     - Constraining and extending
   - Mathematical creating
     - Representing
     - Defining
     - Modifying/transforming/manipulating
   - Integrating strands of mathematical activity
3. **Mathematical work of teaching**
   - Probe mathematical ideas
   - Access and understand the mathematical thinking of learners
   - Know and use the curriculum
   - Assess the mathematical knowledge of learners
   - Reflect on the mathematics of practice

Fig. 2 Brief summary of framework for MPTS

powerful reasoning within a student's otherwise erroneous work. An expert teacher understands the flow of the curriculum of a course and knows how the mathematics in the lesson connects to past lessons and future lessons. An expert teacher knows how to access a student's mathematical thinking and assess how much mathematics the student understands. Expert teachers can reflect on their instruction, analyzing which specific mathematical processes and concepts are understood by their students. This work is mathematical.

Expert teachers need to strive to become proficient in all three categories that are characterized by the framework. Programs that want to increase the mathematical expertise of teachers need to intentionally and explicitly address all three categories of proficiency because the categories are interwoven. In the following study, we investigated teachers' perspectives on the mathematical proficiency useful for teaching mathematics. It provides insight into teachers' perspectives on the potential development of expertise.

3 Studying teachers' perspectives

The Situations were built from classroom mathematical events, but the writing of the Situations and the development of the framework were products of thought and discussions between mathematicians and mathematics educators. Feeling that these final products were now removed from classroom practice, we embarked on a research study to examine the perspectives of Prospective and Practicing teachers on the Situations. In an effort to understand teachers' perspectives on mathematical proficiency, we posed the following research questions: What mathematical ideas do Prospective and Practicing teachers generate when they are presented with the prompt of a situation? How do the ideas generated by Practicing and Prospective teachers compare with those generated by another group of mathematics educators? Is the framework useful in describing the proficiencies identified by Prospective and Practicing teachers?

We found that both the Prospective and Practicing teachers identified mathematical proficiencies that were very similar to those identified by mathematicians and mathematics educators. While this was a validation of the ideas presented in Situations, we were surprised that inexperienced, Prospective teachers identified key proficiencies that were also identified by experienced, Practicing teachers. Since there was such similarity with the original proficiencies of mathematics educators, it is reasonable that the framework was useful in describing the proficiencies identified by teachers and in analyzing their discussions. More results and discussion of this study are available in Wilson and Conner (2009).

Although we were gratified that the study validated the proficiencies and the framework, we had not expected the extent of the teachers' enthusiasm about participating in the study. They enjoyed the mathematical discussions and argued that such discussions were relevant to the practice of teaching mathematics. As we listened to their comments and analyzed their discussions, we began to see the potential for using the Situations for professional learning. In the following sections, we share our methods for employing the Situations with both the Prospective teachers and the Practicing teachers and share some of the insights they provided. We report comments from the teachers that suggest that they believed they were developing their expertise in teaching. Although we had designed the experience as a research study to gather teacher perspectives on useful mathematical knowledge, the participating teachers described the experience as useful professional development. The following sections describe our research protocol, the teachers' reactions to it, and how it may be used as an opportunity for relevant teacher learning.

4 Developing teachers' expertise

Two groups of participants, Prospective teachers and Practicing teachers, worked with the same three Situations, carefully selected from the set of 50 Situations. For both groups, the research sessions were conducted as discussion groups with the instructor as facilitator and two additional researchers observing and taking notes. A modified focus group methodology was used. The goals of the facilitator were to keep the group on the task of identifying mathematical proficiencies that were useful for a given mathematical event and to ensure that all participants had opportunities to contribute to the discussion. If the discussion drifted toward pedagogical issues, the facilitator helped the group return to a discussion of mathematical proficiencies. If the discussion began to focus on *students'* mathematical knowledge, at first, the facilitator restated and later participants reminded each other of the goal of identifying mathematical knowledge that was useful for the *teacher*. The particular Situations used were chosen to represent a range of mathematical content, addressing topics from trigonometry, geometry, and algebra. This mathematical content could be addressed with students at many different grade levels, from grade 6 or 7 to grade 12. In addition, the three situations chosen represented well the kinds of dilemmas or questions presented in the variety of Situations developed for the project.

The discussions of the Situations were structured similarly for both groups. First, the facilitator distributed the "prompt" of the situation, and all participants were asked to read the prompt and think about and write down on an index card what mathematical knowledge would be useful for the teacher in the situation to know. (We specified that this knowledge would not necessarily be ideas to be shared with the students in the situation.) Next, the facilitator asked participants to share their insights and recorded the mathematical ideas on a whiteboard. After asking participants to explain and expand on their initial ideas, the facilitator distributed the foci, one at a time, giving time for the participants to read individually, rate the focus as to how important it was for teachers to know, and then discuss as a large group the mathematical ideas contained therein.

4.1 Case-based learning

As we re-analyzed our data to account for the teachers' beliefs that the Situations were useful for development of teachers' expertise or MPTS, we consulted the literature on case-based learning experiences. Our use of the Situations parallels the use of cases in teacher education. The use of cases for teachers' professional learning has been in vogue since at least the 1980s (Merseth 1996). In mathematics education, case-based teacher education has been used to enhance both Prospective and Practicing teachers' pedagogical and mathematical knowledge (Biza et al. 2007; Lin 2002; Silver et al. 2007; Walen and Williams 2000). The kinds of cases and how they are used vary across different projects and settings (Markovits and Smith 2008). Markovits and Smith (2008) detail two kinds of cases in their review of literature: exemplars, which "exemplify a practice or operationalize a theory" (p. 43) and problem situations, which "provide dilemmas (either mathematical or pedagogical) to be analyzed and resolved" (p. 43). Markovits and Smith describe similar work to ours in the Mathematics Classroom Situations, developed by Markovits and Even for use with elementary and junior high school teachers, in that they give little background information, instead focusing the reader's attention on the problematic situation. In addition, both sets of situations contain possible responses to the situations to which teachers can react. Our Situations, however, are all based on actual classroom episodes and focus the teachers' attention on mathematical knowledge that would be helpful for the teacher in the situation, rather than on pedagogical strategies or how the teacher might interact with the students. When a case describes both pedagogical and mathematical ideas, as in the one described in Markovits and Even (1999), the discussion seems to gravitate more toward the pedagogical notions rather than centering particularly on the mathematical ideas. We wanted to emphasize the importance of teachers having a mathematical proficiency that far exceeds the mathematics they expect of their students.

One commonality in how cases are used across settings and grade levels is the presence of both individual reflection and group discussion. Markovits and Smith (2008) emphasize that "reading a case does not ensure that the reader automatically will engage with all the embedded ideas or spontaneously will make connections to their own practice" (p. 47). In our work with the Situations, the participants were given time to reflect individually on the mathematical ideas before engaging in group discussions. In these discussions, the facilitator revoiced participant contributions and focused the discussions on the mathematical rather than pedagogical aspects of the cases. Case-based learning often occurs in the context of a professional learning community (Steele 2005; Walen and Williams 2000) in which teachers engage in appropriately facilitated discussions. While our research did not include extended time in the context of a professional learning community, our discussions did take on some of the characteristics of a professional learning community, such as establishing norms of communication and developing trust (Borko 2004).

4.2 Methods

In this paper, we report relevant results from our initial analysis of the data gathered related to the mathematical conversations around the three representative Situations. We also report analysis of the data that related to the teachers' perceptions of the Situations as opportunities for developing expertise in teaching mathematics. With this paper, we address the following research questions:

- What were the mathematical ideas discussed by the teachers and how did they fit into the MPTS framework?
- What mathematical ideas did the teachers learn, extend, or recall when engaged in discussions around the Situations?
- What were the perceptions of the participants regarding use of the Situations for professional learning?

Our methodology may be categorized as what Merriam (1998) calls a "basic qualitative study," a form of qualitative research in which the researchers "seek to discover and understand…the perspectives and worldviews of the people involved" (p. 11). In this type of research, analysis relies on identifying patterns, and "findings are a mix of description and analysis" (Merriam 1998, p. 11). While there are no set standards for qualitative research as there are for quantitative research, there have been discussions of what may determine the quality of qualitative research (Freeman et al. 2007). According to Freeman and colleagues, there are multiple ways to assess the quality of qualitative research, but these ways generally include transparency in reporting of research methods and decisions and standards relating to the gathering and presentation of evidence. To preserve the quality of this presentation of our research results, we have detailed our research methods and decisions to the extent possible given the space limitations of a journal. We have also attended to Wilson's (1994) criteria for evidence. Our evidence was "consistent with [our] chosen epistemology or perspective" (p. 26), "observable" (p. 28) to the extent possible by relying on participants' spoken and written words, "gathered through systematic procedures" (p. 29) as we describe below, "shared" (p. 30) to the extent possible by presenting evidence using our participants' own words where possible, and "compelling" (p. 30) in that we present the best evidence we have for each of our results. In short, in conducting and reporting this research, we have adhered to the accepted practices of qualitative educational research, applying them as appropriate to our research situation.

4.2.1 Participants

Prospective teachers. The Prospective teachers included everyone enrolled in a one-semester methods of teaching mathematics course for prospective secondary mathematics teachers. Seven of the Prospective teachers were undergraduates; the eighth was a graduate student enrolled for initial certification. The class sessions in which the research was conducted were structured as discussion groups with the instructor as facilitator and two additional researchers observing and taking notes. One Situation was discussed in each of three class sessions, each of which lasted approximately 90 min. Comments, written and verbal, about the utility of using the Situations were dispersed throughout the sessions.

Possibly because of the amount of time the Prospective teachers had been in classes together and because of the norms negotiated over time between the instructor and the Prospective teachers, the Prospective teachers were comfortable questioning and challenging each other's ideas. In addition, since these discussions occurred in the context of the class, the instructor/facilitator had an additional instructional goal that was not explicitly present in the other group. Thus, the discussions in the Prospective teachers group were more instructional in nature than the discussions in the Practicing teachers group.

Practicing teachers. The second group, the Practicing teachers, comprised six teachers who varied in experience from 3 to 21 years. The Practicing teachers discussed all three Situations in 1 day that was divided into three 90-min sessions. The researcher who taught the methods course also facilitated all discussions with the Practicing teachers; the same researchers took field notes of each discussion. The three discussions were our primary data source for this paper, but we also used data from an additional discussion, prompted by the teachers after all Situations had been discussed, which addressed the use of the Situations for professional learning.

The Practicing teachers came to the campus to participate in research designed to describe MPTS. They were asked to participate in research as knowledgeable persons about MPTS, and this was not characterized as a learning experience. In addition, while some of the teachers knew each other prior to their participation, most were not acquainted with each other, and so it took some time (during the discussion of the first Situation) to become comfortable with challenging each other and building on each other's responses.

4.2.2 Data and analysis

All written work of the participants, including their initial reactions to the prompts and their ratings of the foci were collected. The data corpus included six 90-min audio and video recordings of the sessions, one 30-min audio and video recording of the teacher-prompted discussion of the utility of the Situations for professional learning, two sets of notes for each session, and 14 sets of written work. The participants' written work was transcribed and organized into a chart for each session. To analyze the data, we individually watched the videos, augmenting with the audio recordings, and noted what the participants stated. Our unit of analysis was the group and not the individual participant. We listed the statements of the participants and compared our notes, resolving differences by watching the videos together. When we had an accurate record of each discussion (as demonstrated by an agreed-upon list of the mathematical concepts stated by the group in the first part of the discussion and agreement as to the inferred meanings of explanatory statements in the second part of the discussion), we searched for themes in the data, working individually and then coming together to compare and resolve differences. This was an iterative process in which we searched for confirming and disconfirming evidence of all suggested themes, rejecting some and keeping the ones for which we found evidence. Themes involved mathematical ideas that could often be somewhat separated from the specific mathematical area, such as valuing definitions and properties, as well as specific mathematical knowledge expressed, such as knowing how to circumscribe a triangle. We finally compiled our themes across groups and Situations, enumerating the evidence for each within the document.

After completing our analysis of the teachers' mathematical statements with respect to the foci and framework, we looked back at the data to consider our participants' statements with respect to the utility of the Situations for professional learning. We identified some episodes in which participants' views of mathematical ideas seemed to change, as inferred from changes in their discourse about the ideas. We also identified statements from the participants about the utility of the discussion itself and the potential utility of use of the Situations in professional development settings (for both Prospective and Practicing teachers).

5 Results

As we examined the Practicing and Prospective teachers' responses to the prompts of the Situations presented to them, we found that their responses echoed many aspects of the framework for Mathematical Proficiency for Teaching. We also found the Situations to be useful tools for facilitating mathematical discussions that reinforce and develop important mathematical proficiencies for teaching. The Situations helped the instructor (facilitator) access what the participants thought was important mathematics; the foci in particular directed the participants' attention on some relevant mathematical ideas, providing alternative ways for thinking about the mathematics in the Situations; and the foci provided a higher level of rigor within what was otherwise a less formal mathematical discussion. Finally, the combination of the brief description with the intentionally mathematically focused questions allowed the participants to discuss mathematical rather than only pedagogical ideas, which they were often more comfortable discussing. Although teachers often used mathematics-specific pedagogical ideas as rationales for their statements, the primary focus remained the mathematical ideas that were contributed. Comments from the participants and observations of the researchers suggest that the Situations could be used in fruitful professional development for teachers.

We organize our presentation of results around the framework for MPTS, noting that the teachers mentioned ideas relating to all three categories of the framework as well as many of the subcategories. We limit our discussion of the results to one example of each of the categories and conclude with a description of how our participants suggested these Situations were useful for learning mathematics and would be useful for continuing professional development. We take the fact that the teachers' discussions included each category of the framework as evidence that mathematical discussions provoked by the Situations address a wide range of mathematical ideas.

5.1 Strategic competence, an example of mathematical ability

One of the Situations presented to our participants involved an event in which inverse trigonometric functions were confused with reciprocal trigonometric functions (that is, a student teacher's plan named the secant, cosecant, and cotangent functions as inverse trigonometric functions). As the Prospective teachers were discussing the Situation, they indicated that they valued using multiple strategies for solving problems. In response to the third focus, which presented an argument involving reflecting the graph of $y = csc\ x$ over the line $y = x$ and comparing that to the graph of $y = sin\ x$, one of the Prospective teachers said, "It's just another way of understanding inverses. Some people are going to understand it better algebraically, others graphically, so it's important for the teacher to understand it both ways" (Prospective Teachers Discussion 1). Another suggested, "Students learn material in different ways. Being able to teach various ways to teach different concepts is something that a teacher should know, just like she said. Students might learn things graphically better than they do algebraically" (Prospective Teachers Discussion 1). Although several students preferred the previous, more symbolic, arguments, the general consensus was that having multiple strategies to solve a problem was important, first for the teacher, but also for students. This is consistent with the category of strategic competence under mathematical ability, which is described as requiring "the ability to generate, evaluate, and implement problem-solving strategies" (Wilson and Heid 2010, p. 7).

As the Practicing teachers discussed the same Situation, they talked about the importance of presenting students with problems whose solutions would require applications of the reciprocal trigonometric functions and the inverse trigonometric functions. In particular, they mentioned presenting applied problems in which students would need to recognize the need for one type of function or the other, "If we could talk about situations in which you would use the inverse trig function versus situations in which you would use the reciprocal, because then, if you make it a little more real, I think it just gives more motivation to really understand it and you see why there is a need for the difference" (Practicing Teachers Discussion 3). The Practicing teachers clearly valued students' abilities to go beyond simply graphing or recognizing reciprocal and inverse trigonometric functions and to flexibly work with either one in appropriate ways in problem-solving situations, another demonstration of their valuing of the strategic competence aspect of mathematical ability. One of them also stated, "It would be helpful [to know more than one approach] because I could see students coming up with all different approaches…we still need to know others, so I could help students with whatever they may come up with. They need to know that one approach isn't always the best" (Practicing Teachers Discussion 3).

5.2 Reasoning involving constraining and extending, an example of mathematical activity

One aspect of mathematical activity that emerged in several of the discussions was the idea of constraining and extending a mathematical entity. This fits into the framework as a subcategory of mathematical reasoning. For instance, when discussing the Circumscribing Polygons Situation, the Prospective teachers were concerned about the generalizability of the focus that discussed cyclic quadrilaterals (see Focus 2 of Fig. 3), saying, "I don't think you can extend this to [the situation of a polygon with more sides] because it's talking about opposite angles, and in a hexagon, which one's opposite?" (Prospective Teachers Discussion 2). Even before reading the foci for the Circumscribing Polygons Situation, the Practicing teachers were concerned with the possibility of over-generalizing: "Just because it works for triangles, we don't want them to assume it works for every polygon" (Practicing Teachers Discussion 2). As each group continued to discuss the Situation, they emphasized the importance of knowing when (and when not) to apply particular ideas.

Likewise, from the Exponents Situation (see Fig. 1) arose the importance of how a domain is restricted. When discussing the Exponents Situation, both groups pointed out that there were no explicit restrictions on x, m, or n in the given statement of the problem, and depending on whether x or m and n are restricted, the other variable can take on different values. The foci of the Situation were written with the assumption that the entire expression was limited to the real numbers. Both groups wished to extend the range of the expression as well as the domain of each variable to the complex plane, although they encountered certain difficulties with doing so. In the end, each group reached consensus that restricting the values in this Situation to real numbers (and thus restricting individual values even further) made sense in a secondary teaching situation.

5.3 Know and use the curriculum in relation to learners, an example of the mathematical work of teaching

While we did not expect to find much evidence of the mathematical work of teaching in a focus group setting where the purpose was not to facilitate mathematical learning, Practicing and Prospective teachers talked about understanding what students know as something that was important to them in preparing to answer students' questions. In the Inverse Trigonometric Functions Situation, the Prospective teachers started their discussion by asking, "What do the high school students already know about trig? Are you just throwing sine and cosine at them or do they already know about trig in triangles?" (Prospective Teachers Discussion 1). When examining the second focus of the Exponents Situation (see Fig. 1), one of the Practicing teachers was concerned about the mathematical background of students, saying, "I don't think they've had the math knowledge up to that point to be able to follow this" (Practicing Teachers Discussion 1). Another teacher reminded her that their focus was on what was helpful for the teacher to know, but the teacher's concern for appropriately addressing the students' prior knowledge was apparent.

5.4 Utility of building on mathematical events for teachers' learning

Our participants' own words, and our observations of the participants, give credence to the idea that discussions such as these that build on classroom events are useful tools for teacher learning. Both groups of participants' conversations were punctuated by occasional statements about this being a novel way to think about these mathematical ideas, such as, "I just never really thought about it that way before" (Prospective Teachers Discussion 3) or "I've never discussed nonsequential polygons before" (Practicing Teachers Discussion 2). As we worked through the Situations, the researchers noted that, particularly for the

Fig. 3 Condensed version of Circumscribing Polygons Situation

Situation 43: Circumscribing Polygons

Prompt

In a geometry class, after a discussion about circumscribing circles about triangles, a student asked, "Can you circumscribe a circle about any polygon?"

Commentary

A polygon that can be circumscribed by a circle is called a cyclic polygon. Not every polygon is cyclic, but there are infinite cyclic polygons. This can be understood by considering a given circle and all the possibilities of how many points can be placed on the circle, and then connected to form a polygon. However, there are certain classes of polygons that are noteworthy because they are always cyclic. The conditions under which a circle circumscribes a given polygon are dependent upon the relationships among the angles, the sides, and the perpendicular bisectors of the sides of the polygon. The following foci describe classes of cyclic polygons in order of the number of their sides: triangles, certain quadrilaterals, and regular polygons. Focus 3 provides one way to check whether a given polygon is cyclic: a polygon is cyclic if and only if the perpendicular bisectors of all of its sides are concurrent. Though the inclusion of various geometries would provide interesting discussion, the Foci in this Situation are limited to Euclidean geometry in a plane.

Mathematical Focus 1

Every triangle is cyclic. This fact is core to establishing a condition for other polygons to be cyclic.

Mathematical Focus 2

A convex quadrilateral in a plane is cyclic if and only if its opposite angles are supplementary.

Mathematical Focus 3

There are four-sided figures in the plane that behave differently from convex quadrilaterals. Concave quadrilaterals are never cyclic, and a four-sided figure with non-sequential vertices is cyclic if and only if its "opposite" angles are congruent.

Mathematical Focus 4

Every planar regular polygon is cyclic. However, not every cyclic polygon is regular.

Prospective teachers, the participants changed the ways they talked about the mathematics, building on other participants' observations, and changing or solidifying their understanding of the mathematical ideas.

After the Practicing teachers had finished discussing the Situations, we described the research project that led to the development of the Situations and asked for any questions they had. Instead of asking questions about the research, the Practicing teachers engaged in a discussion of how they had learned mathematics through the discussion of the Situations. One of the Practicing teachers articulated his appreciation for the kind of learning he experienced in the focus group.

> Maybe the approach that mathematics that's taught in education schools should be a little bit different in that like I think the activities we just went through, I was thinking, "Wow, I've learned a lot and I'm going to change the way, like especially this last one, I'm going to change the way I teach it, and I learned some new math." I just wonder if an approach where you do this type of discussion and then maybe before the discussion you teach some of the formal group theory stuff that we kind of, that was kind of addressed in focus 1. I thought I just learned a lot. And that the discussion we just had was more effective than a lot of the stuff I did in math in my master's degree classes. (Teacher 5)

Multiple Practicing teachers recounted aspects of the discussion that they found to be useful. In general, they said that they would find this kind of professional development activity to be useful for teachers. "I think that [it] would be helpful to have this kind of thing as a professional development" (Teacher 6). In addition, they mentioned how they saw the usefulness of mathematical knowledge that they already had in different ways. "I've learned a lot. And this is extremely useful stuff I can take back and, you know, use in the classroom cause this is stuff that we've been doing as teachers and it's just some things that when we teach this material we don't necessarily think of all of these things" (Teacher 3).

The Prospective teachers said that they found the Situations to be one of the most helpful aspects of their methods course (in which they examined cases, planned lessons, and discussed readings about teaching). One Prospective teacher said, "I like to think about this stuff" (Prospective Teacher K); another, "I wish we had done it more often" (Prospective Teacher B); a third, "They're my favorite thing" (Prospective Teacher D). The instructor was originally a bit hesitant to use the Situations in the course, fearing the students would not see the relevance. By the end of the course, both the instructor and the students (Prospective teachers) were convinced of the relevance and utility of the Situations for discussing important mathematical ideas for teaching.

5.5 Limitations of the study

Our use of the Situations was primarily intended to be a research tool in developing a framework for mathematical proficiency in teaching, but we believe that the teachers were sincere in their claims about developing mathematical understanding from the discussion. While we are convinced that the Situations are useful tools for engaging in mathematical discussions that may lead to developing greater expertise in teaching, much depends on the environment in which they are used, the teachers involved, and the way they are introduced. In addition, in using the Situations for professional learning, it might be important to choose Situations that address mathematical ideas at a level of sophistication that is appropriate for a particular group.

Because of the purpose of our study and the way in which it was designed, we can make no claims about changes in our participants' practice. Our insights are based on participants' statements within a focus group setting, and not on observations of their teaching. We do not claim that our findings would generalize to any other group; rather, we propose that our participants found this to be a useful exercise and they suggested that others would also do so.

6 Implications

As we reflect on how our use of the Situations resulted in productive discussions about mathematics and ultimately in teachers learning about mathematics, we identified three actions that we believe were key in our implementation. First, we established a non-threatening environment in which to access the participants' mathematical knowledge. Second, we engaged participants in discussing the prompt first, then, after a reasonable time for discussion, introduced the foci, discussing each focus by itself and then all foci as a group, comparing their relative usefulness. Third, each time the discussion began to turn to pedagogy, we gently refocused the discussion on mathematical ideas.

The Situations themselves assisted in establishing a non-threatening environment. The prompts, being set in real classrooms, resonated with the teachers as questions they could imagine (or had experienced) students asking. We asked the teachers what knowledge a teacher would find useful in the Situation, making it clear that we valued their insights as teachers. Creating a list of mathematical ideas, in the spirit of brainstorming, did not require teachers to reveal deficits in their mathematical knowledge. During the subsequent discussion of the brainstormed ideas, teachers sometimes revealed gaps in knowledge that led to more and deeper discussions of some of the mathematical ideas. For example, in the discussion of the prompt of the Situation described in the opening paragraph of this paper, the Prospective teachers brainstormed a list of ideas that included "which polygons can be circumscribed." During the subsequent discussion, a Prospective teacher hypothesized that regular polygons could be circumscribed, but 'irregular' ones could not. A different Prospective teacher offered a trapezoid as a counterexample to that hypothesis, demonstrating that the particular trapezoid considered was not a regular polygon and yet could be circumscribed. During this part of their discussion, they defined polygon and regular polygon, described how to circumscribe a triangle, and listed characteristics of perpendicular bisectors. For some students, such as the one who thought polygons had to be regular to be circumscribed, this discussion involved new mathematical ideas. For others, it may have involved making connections between mathematical ideas that were familiar to them (such as the characteristics of a perpendicular bisector and the definition of a circle).

Engaging participants in a discussion of the prompt before introducing the foci allowed us to access their mathematical ideas about a situation before introducing our own. Subsequently introducing the foci allowed some participants to revise and extend their mathematical understandings. For instance, when discussing the third focus of the Inverse Trigonometric Functions Situation, one of the Practicing teachers mentioned that while she had asked students to verify that two functions were inverses by comparing their graphs on the same plane as the line $y = x$, she had never considered using a graph as a counterexample as is demonstrated in the focus (see Fig. 4). She said, "I've used this for other functions, but never really thought of it as using it as a counter-example… I've used it for when they do just other graphs and the other functions and it's like they're checking to see if they see the reflection in the line $y = x$ … but I really like this to use with the trig functions" (Practicing Teachers Discussion 3). Sometimes, participants remarked upon knowledge that others brought up. For instance, when discussing the prompt of the

Fig. 4 Focus 3 of the Inverse Trigonometric Functions Situation

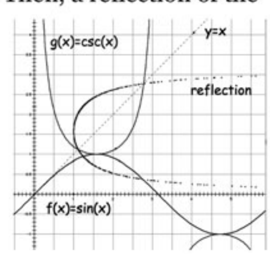
Mathematical Focus 3

When functions are graphed in an xy*-coordinate system with* y *as a function of* x*, these graphs are reflections of their inverses' graphs (under composition) in the line* y = x.

The graph of a function reflected in the line $y = x$ is the graph of its inverse function, though without restricting to principal values, the inverse may not be a function. Justifying this claim requires establishing that the reflection of an arbitrary point (a, b) in the line $y = x$ is the point (b, a).

The following argument shows that the reflection of an arbitrary point (a, b) in the line $y = x$ is the point (b, a), using the geometric properties of reflection (the point A' is a reflection of the point A in line l if and only if l is the perpendicular bisector of $\overline{AA'}$). First, we note that the perpendicular bisector of $\overline{AA'}$ consists of precisely those points in the plane that are equidistant from A and A'. So, we choose an arbitrary point on the line $y = x$, say (c, c). Using the distance formula to compute the distance between (a, b) and (c, c) and between (b, a) and (c, c) we find that $\sqrt{(c-a)^2 + (c-b)^2} = \sqrt{(c-b)^2 + (c-a)^2}$ and so the line $y = x$ is the perpendicular bisector of the line segment between (a, b) and (b, a). Therefore, the reflection of the point (a, b) in the line $y = x$ is (b, a).

Suppose that cosecant and sine were inverse functions. Then, a reflection of the graph of $y = \csc(x)$ in the line $y = x$ would be the graph of $y = \sin(x)$. The figure at the right shows, on one coordinate system, graphs of the sine function, the line given by $y = x$, the cosecant function, and the reflection of the cosecant function in the line $y = x$. Because the reflection of the graph of the cosecant function in the line $y = x$ does not coincide with the graph of the sine function, sine and cosecant are not inverse functions.



Exponents Situation, one Practicing teacher remarked, "If x is zero, then m and n cannot be zero, because zero to the zero power is indeterminate." Another teacher responded, "That's an interesting point, because I don't teach calculus, so I've forgotten all about that. That's teacher knowledge that I didn't really have, that could have been useful" (Practicing Teachers Discussion 1). The participants learned, or were reminded of, different mathematical ideas through the conversations about the prompts and the foci.

At some point, most discussions of the Situations began to drift toward discussion of pedagogy. Since our goal was to discuss mathematical ideas, we refocused discussions that began to drift by building on the pedagogical idea to ask what mathematical knowledge would be useful to enact the pedagogical idea. Eventually, our participants began to focus each other's attention on mathematics rather than pedagogy.

7 Concluding thoughts about using Situations for teacher learning

We found that engaging teachers in discussions of Situations arising from classroom events prompted rich mathematical discussions that were valued by the participating teachers. Using a focus group format and basic qualitative analysis, we gained insight into our research questions and were able to document a professional development experience that focused teachers' conversations on mathematics. We have shared this experience with two purposes in mind. We offer the research protocol as a procedure that can be used to engage teachers in mathematical discussions that are relevant to their teaching and have the potential to increase teachers' MPTS and thus their expertise. Since we worked with both prospective mathematics teachers and practicing mathematics teachers, we argue that similar activities can benefit a wide range of mathematics teachers. Our second purpose is to present and share a framework describing MPTS. We hope that others find this framework helpful as they plan professional development, create materials, or begin to assess the relevance of their work for building teachers' mathematical expertise that is useful in the classroom.

Our first research question investigated teachers' perspectives about the mathematical expertise that was useful in their teaching. Our analysis indicated that both Prospective and Practicing teachers identified mathematical concepts that were aligned with the framework for MPTS developed by mathematicians and mathematics educators. The fact that educators with varied backgrounds identified similar mathematical ideas leads us to believe that our framework is useful in specifying important aspects of mathematical proficiency.

Our second research question explored what mathematical ideas teachers claimed to learn, extend, or revisit as part of the discussion. The teachers in our study may have reflected on many of the mathematical ideas discussed in the focus groups, and we do not claim to have captured the extent of their learning in these situations. However, the teachers, within their discussions, pointed out mathematical ideas that they had not thought about recently or had never in their memory encountered that they thought would be useful as they taught mathematics in the future. We did not formally inquire into what the participants learned by using a pretest and posttest or even asking them to respond to a survey. Our conclusions regarding their learning are based on their own comments within the discussions; while these do not portray the entire picture, they appear to represent a subset of the kinds of learning (and thus the development of expertise) that may occur during these kinds of discussions.

Our third research question helped us to understand how to engage teachers in important mathematical discussions. Teachers were able to share information with each other that not only encouraged engagement, but also helped teachers see the relevance of advanced mathematics to the mathematics they were teaching. Based on comments from participants, we are convinced that both Prospective and Practicing teachers found the Situations useful for their own learning and professional development. They found comfortable ways to discuss the mathematics by offering examples of mathematics knowledge that they found useful in response to the situations posed by the researchers. They could offer basic content knowledge such as teachers should "be familiar with what sine, cosine, tangent, secant, cosecant, cotangent look like on a graph" (Prospective Teachers Discussion 1), or they could question their colleagues about advanced concepts such as the impact of extending the domain of exponents to include complex numbers. We found that they built on each other's comments, which allowed those who had not taught a topic recently to refresh their proficiency and thus build their expertise.

Since we did not visit classrooms, we would like to learn more about how teachers used these discussions. The next few steps would include seeing how teachers used their new MPTS. We would like to know if teachers who discussed the Situations in a professional learning environment would demonstrate their developed expertise by building on mathematical events in their own classrooms.

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