MAC-CPTM Situations Project

Situation 46: Division Involving Zero

Prompt

On the first day of class, preservice middle school teachers were asked to evaluate $\frac{2}{0}$, $\frac{0}{0}$, and $\frac{0}{2}$ and to explain their answers. There was some disagreement among their answers for $\frac{0}{0}$ (potentially 0, 1, undefinable, and impossible) and quite a bit of disagreement among their explanations:

- Because any number over 0 is undefined;
- Because you cannot divide by 0;
- Because 0 cannot be in the denominator;
- Because 0 divided by anything is 0; and
- Because a number divided by itself is 1.

Commentary

The mathematical issue centers on the possible values that result when zero is the dividend, the divisor, or both the dividend and the divisor in a quotient. The value of such a quotient would be zero, undefinable, or indeterminate, respectively. The foci use multiple contexts within and beyond mathematics to represent and illustrate these three possibilities. Connections are made to ratios, factor pairs, Cartesian product, area of rectangles, and the real projective line.

Mathematical Foci

Mathematical Focus 1:
An expression involving real number division can be viewed as real number multiplication, so an equation can be written that uses a variable to represent the number given by the quotient. The number of solutions for equations that are equivalent to that equation indicate whether the expression has one value, is undefinable, or is indeterminate.

We can think of a rational number as being the solution to an equation. If division expressions involving zero also represent rational numbers, we should have consistent results when we examine equations involving these expressions. To find the solution of the equation $\frac{0}{2} = x$, we consider the equivalent statement $2x = 0$, which yields the unique solution $x = 0$. To see the impossibility of a numerical value for a rational number with a 0 in the denominator, we consider the equation $\frac{0}{0} = x$, and its potentially equivalent equation, $0x = 0$. Because any value of *x* is a solution to this equation, there are infinitely many solutions; hence, no unique solution, and so the expression $\frac{0}{0}$ is indeterminate. With the same thinking, if $\frac{2}{0} = x$, then $0x = 2$. No real number *x* is a solution to this equation, and so the expression $\frac{2}{0}$ is undefinable.

Mathematical Focus 2

One can find the value of whole-number division expressions (with no remainder) by finding either the number of objects in a group (a partitive view of division) or the number of groups (a quotitive view of division).

In partitive division, we take a total number of objects and divide the objects equally among a number of groups. A non-zero example would be $\frac{12}{3}$, where we share 12 objects equally among 3 groups and ask how many objects would be in one group. Similarly, $\frac{0}{2}$ can be thought of as 0 objects in 2 groups, which means 0 objects in each group. Additionally, the expression $\frac{0}{0}$ is a model for dividing 0 objects among 0 groups. In other words, "If 0 objects are shared by 0 groups, how many objects are in 1 group?" There is not enough information to answer this question, and so the expression $\frac{0}{0}$ is indeterminate. If the number of objects in a group is 3, or 7.2, or any size at all, 0 groups would have 0 objects. Similarly, $\frac{2}{0}$ is a model for the example: "If 2 objects are shared by 0 groups, how many objects are in 1 group?" In this case the number of objects in the group is undefinable, because there are 0 groups.

Using a quotitive view of division, we interpret the expression, $\frac{12}{3}$ as a model of splitting 12 objects into groups of 3 and asking how many groups can be made. So $\frac{0}{2}$ can be thought of as splitting 0 objects in groups of 2, which means 0 groups of size 2. The expression $\frac{0}{0}$ models the splitting of 0 objects into groups of size 0, and asks how many groups can be made. Because there could be any number of groups, there are an infinite number of solutions, and so the expression is indeterminate. Lastly, the expression $\frac{2}{0}$ models the splitting of 2 objects into groups of 0, and asking how many groups can be made. Regardless of how many groups of 0 we remove, no objects are removed. Therefore, the number of groups is undefinable.

Mathematical Focus 3

The mathematical meaning of a/b *(for real numbers* a *and* b *and sometimes, but not always, with* b ≠ 0*) arises in several different mathematical settings, including: slope of a line, direct proportion, Cartesian product, factor pairs, and area of rectangles. The meaning of* a/b *for real numbers* a *and* b *should be consistent within any one mathematical setting.*

There are mathematical situations in which ratios are necessary, and a quotient can be reinterpreted as a ratio. For example, the slope of a line between two points in the Cartesian plane can be defined as the ratio of the change in the y-direction to the change in the x-direction, or as the rise divided by run. In the case of two coincident points, the change in the y-direction and the change in the x-direction are both 0, which means that the rise divided by run is $\frac{0}{0}$. There are an infinite number of lines through two coincident points, and so the slope is indeterminate. In the case of two points lying on the same vertical line whose y-coordinates differ by a, the change in the y direction will be a and the change in the x direction is 0. It might be tempting to claim that since the slope of a vertical line is undefinable, that $\frac{a}{0}$ is undefinable. However, this claim is exactly what we are trying to show.

The model for direct proportion, $y = kx$, suggests a family of lines through the origin. For y and non-zero x as the coordinates of points on a line given by $y=kx$, the ratio $\frac{y}{x}$ equals k, which is constant. If this ratio held for the coordinates of the origin, it would be $\frac{0}{0} = k$. However, no one value of k would make sense as the value of $\frac{0}{0}$ because the origin is on every line represented by an equation of the form $y = kx$. Thinking about the equation $y = kx$ in terms of number relationships also leads to the conclusion that the value of $\frac{0}{0}$ cannot be determined: if $y = kx$ and $x = 0$, then $y = 0$ and k can be any real number, just as in Focus 1. It is important to note that in the case where $x=0$ and $y \neq 0$, such as $\frac{2}{0}$, it is difficult to explain via direct proportion; if $y = kx$, then $x=0$ and $y \neq 0$ is an impossible circumstance.

A different mathematical context for looking at division involving zero is the Cartesian product. A non-zero example is this: if 12 outfits can be made using 3 pairs of pants and some number of shirts, how many shirts are there? There must be 4 shirts, as this would give 12 pants/shirt combinations. Similarly, if 0 outfits can be made using 2 pairs of pants and some number of shirts, there must be 0 shirts. If 0 outfits can be made using 0 pairs of pants and some number of shirts, the number of possibilities for the number of shirts is infinite. Lastly, how many shirts are there if there are two outfits and 0 pairs of pants? No possible number of shirts can be used to make 2 outfits if there are 0 pairs of pants.

In the context of factor pairs, a division expression with an integral value represents an unknown factor of the dividend. For $\frac{12}{3}$, 3 and the quotient are a factor pair for 12. In this expression, 12 can be written as the product of 3 and the quotient: $12 = 3 \times 4$. For $\frac{0}{2}$, 2 and the quotient are a factor pair for 0. Therefore, the quotient must be 0, because $0 \times 2 = 0$. For $\frac{0}{0}$, 0 is part of an infinite number of factor pairs for 0 and so the expression is indeterminate. For $\frac{2}{0}$, 0 is not part of any factor pair for 2, thus the expression is undefinable.

One side length of a rectangle is the quotient of the area of the rectangle and its other side length. Suppose we allow that rectangles can have side lengths of 0. If a rectangle has area 12 and height 3, what is its width? It would be a width of 4. If a rectangle has area 0 and length 2, its width is 0 and so 0 divided by 2 is 0. If a rectangle has area 0 and height 0, what is its width? Any width is possible and so 0 divided by 0 is indeterminate. If a rectangle has area 2 and height 0, what is its width? It is impossible for a rectangle to have area 2 and height 0 and so 2 divided by 0 is undefinable.

Mathematical Focus 4:
Contextual applications of division or of rates or ratios involving 0 *illustrate when division by* 0 *yields an undefinable or indeterminate form and when division of* 0 *by a non-zero real number yields* 0.

If Angela makes 3 free throws in 12 attempts, what is her rate? If Angela makes 0 free throws in 2 attempts, her rate is 0. If Angela makes 0 free throws in 0 attempts, her rate could be any of an infinite number of rates. On the other hand, since it is not possible for Angela to make 2 free throws in 0 attempts, it is not possible to determine her rate.

Determining the speed of an object over a given period of time is another rate context. If one goes 12 miles in 3 hours at a constant speed, how fast is one going? The answer is 4 miles per hour. If one goes 0 miles in 2 hours at a constant speed, one is going 0 miles per hour. If one goes 0 miles in 0 hours, how fast is one going? An infinite number of speeds are possible. If one goes 1 mile in 0 hours, how fast is one going? This travel situation is an impossible setting. [Note that there is a sense of infinite speed here, so it might be tempting to define $\frac{1}{0}$ as infinity. However, this leads to further complications, as in Focus 5.]

Additionally, the idea of rate is prevalent when discussing the unit price, as when purchasing multiple quantities of an item in a store. If $12 buys 3 pounds of tomatoes, how much is 1 pound? If $0 buys 2 pounds of tomatoes, then 1 pound can be bought for $0. If $0 buys 0 pounds of tomatoes, there is an infinite number of possible costs for 1 pound.

Mathematical Focus 5:
Slopes of lines in two-dimensional Cartesian space map to real projective one-space in such a way that confirms that the value of a/b *when* b = 0 *is undefinable if* a ≠ 0 *and indeterminate if* a = 0.

In the Cartesian plane, consider the set of lines through the origin, and consider each line (without the origin) to be an equivalence class of points in the plane.

Except when $x = 0$, the ratio of the coordinates of a point gives the slope of the line that is the equivalence class containing that point. The origin must be excluded because it would be in all equivalence classes, which is rather like saying that $\frac{0}{0}$ would be the slope of any line through the origin [see Focus 3]. Note that the slope of a line through the origin is equal to the y-coordinate of the intersection of that line and the line $x = 1$. This way, we can use slope to establish a natural one-to-one correspondence between the equivalence classes (except for the equivalence class that is the vertical line, since it does not intersect the line $x = 1$) and the real numbers. Thus, the real numbers give us all possible slopes, except for the vertical line.

When $x = 0$, all the points in the equivalence class lie on the vertical line that is the y-axis. (Again the origin must be excluded from this equivalence class.) The ratio of the coordinates is undefinable, so the slope is undefinable. As positively sloped lines approach vertical, their slopes approach ∞, suggesting the slope of the vertical line to be ∞. As negatively sloped lines approach vertical, their slopes approach –∞, suggesting the slope should instead be –∞. However, there is only one vertical line through the origin, so it cannot have two different slopes. To resolve this ambiguity, we can decide that ∞ and –∞ are the same "number" because they should represent the same slope. So now, if we think about all possible slopes, we have all real numbers and one more number, which we will call ∞. Imagine beginning with the extended real line, $\mathbb{R} \cup \{\infty, -\infty\}$, and gluing together the points ∞ and –∞ so that they are the same point. This is the real projective one-space, $\mathbb{R} \cup \{\infty\}$.

Post-Commentary

For situations involving division with zero, there are three types of forms: 0, undefinable, and indeterminate. The indeterminate form has particular importance in a calculus setting in that: given a function, *f*, that would be continuous everywhere except that *f*(*a*) is indeterminate, we can select a functional value to make a related function that is continuous everywhere. For all of its domain values except *a*, the new function would have the same values as the given function. For example, in the case of the function $f(x) = \frac{\sin x}{x}$, the function is continuous for all real numbers except 0, for the functional value at *x* = 0 is the indeterminate form $\frac{0}{0}$. The piecewise-defined function, $f(x) = \begin{cases} \frac{\sin x}{x}, x \neq 0 \\ 1, x = 0 \end{cases}$ is continuous for all real numbers. In this case, we used the fact that the limit of interest was 1: $\lim_{x \to 0} \frac{\sin x}{x} = 1$. However, in other cases, limits related to $\frac{0}{0}$ do not have to be 1, or even an integer. For example, $\lim_{x \to 0} \frac{2 \sin x}{3x} = \frac{2}{3}$. These are but two examples that show that, depending on the function, we would find it useful to assign two different numerical values for a limit involving $\frac{0}{0}$. The very ambiguity of this form suggests the need for L'Hospital's rule.