MAC-CPTM Situations Project

Situation 40: Powers

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Prompt

During an Algebra I lesson on exponents, the teacher asked the students to calculate positive integer powers of 2. A student asked the teacher, "We've found 2^2 and 2^3. What about $2^{2.5}$?"

Commentary

The student's simple question was important because it extended the domain of the exponent from integers into rational numbers. The foci explore the nature of exponents numerically, analytically, and graphically. They illustrate that exponentiation is more than repeated multiplication and clarify the definition and properties of exponents..

Consistency is maintained in all three foci. The choices that are made in the extension of the definition and properties of exponents are made so that they are preserved in a different number system, i.e. from whole numbers to rational numbers. This need for consistency permeates the entire Situation and is lying underneath the surface of each Focus. For example, in Focus 2, we consider how we can define 2^{rational} in a way that allows the properties of exponents to hold. By deciding that the properties must hold, we can then use them to define the n-th root of 2 as $2^{1/n}$.

It is also useful to consider the fact that $f(x) = a^x$ would behave differently for particular values of "a" and "x." In the discussion below, "a" is assumed to be positive and "x" is a rational number. If "a" had been negative, the arguments would not apply. Furthermore, different mathematical discussions would be necessary if "x" had been irrational, transcendental, or complex. See Situation 21 for a further discussion of this.

Mathematical Foci

Mathematical Focus 1

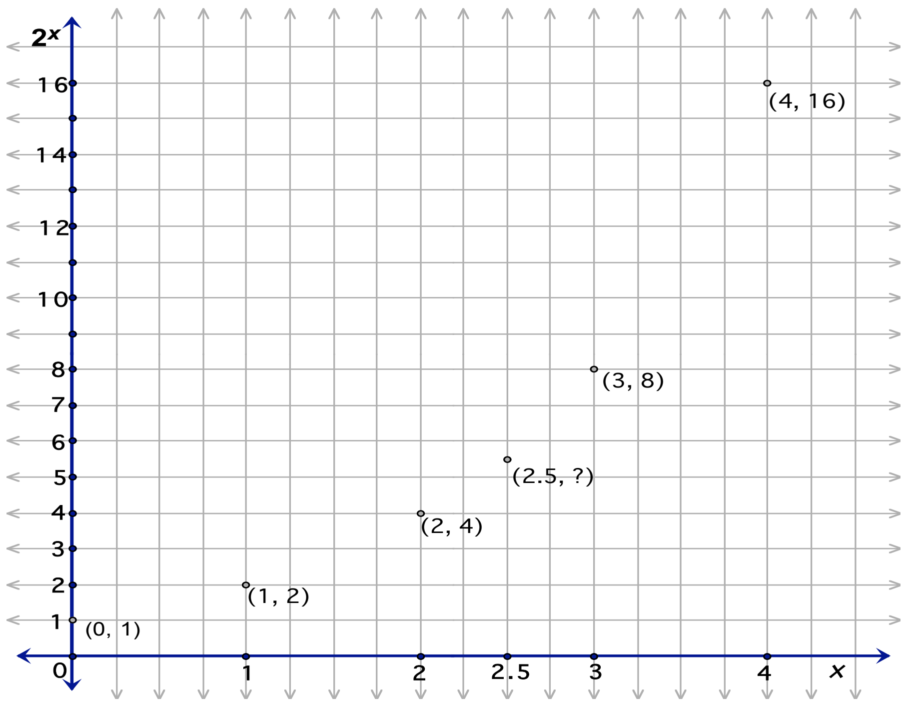
The value for the unknown value $2^{2.5}$ can be estimated based on the known values for 2^2 and 2^3.

It is important to understand that the value for $2^{2.5}$ will not be halfway between 2^2 and 2^3. The table below shows a pattern of increasing growth between successive values of 2^x and illustrates that exponential growth is different from a linear pattern of growth. Specifically, nonlinearity implies that even though 2.5 is the average of 2 and 3, $2^{2.5}$ will not be the average of 4 and 8.

x	2^x
0	1
1	2
2	4
3	8
4	16

Since the *differences* between the successive values of 2^x are increasing, one can argue that linear approximations will overestimate the value of $2^{2.5}$. Therefore, $2^{2.5}$ should be less than 6.

Visualizing this numerical information is also helpful to see why the value of $2^{2.5}$ will be closer to 4 than to 8.



Specifically, if we look at this graph, we can begin to think of a way to connect the points that preserves the pattern. This pattern of points creates a graph that is concave up. We can use this fact to conclude that the value of $2^{2.5}$ will be closer to 4 than to 8.

Mathematical Focus 2

Properties of exponents suggest a particular value for $2^{2.5}$.

The value for $2^{2.5}$ can be explored using properties of exponents. The expression $2^{2.5}$ can be rewritten as $2^2 \bullet 2^{0.5} = 2^2 \bullet 2^{1/2}$ or $2^{5/2}$.

Before we use $2^{1/2}$ in this focus, it is important to define $2^{1/n}$. Using the properties of exponents, we know that $2^{1/n} \bullet 2^{1/n} \ldots 2^{1/n} = 2^{n/n} = 2^1 = 2$. Furthermore, $2^{1/n} = \sqrt[n]{2}$. Specifically, $2^{1/2} = \sqrt{2}$. Using this representation with the properties of exponents, this quantity can be represented by

$2^2 \cdot 2^{0.5} = 2^2\sqrt{2} \approx 4(1.414) = 5.656$. So $2^{2.5} \approx 5.656$. Similarly,

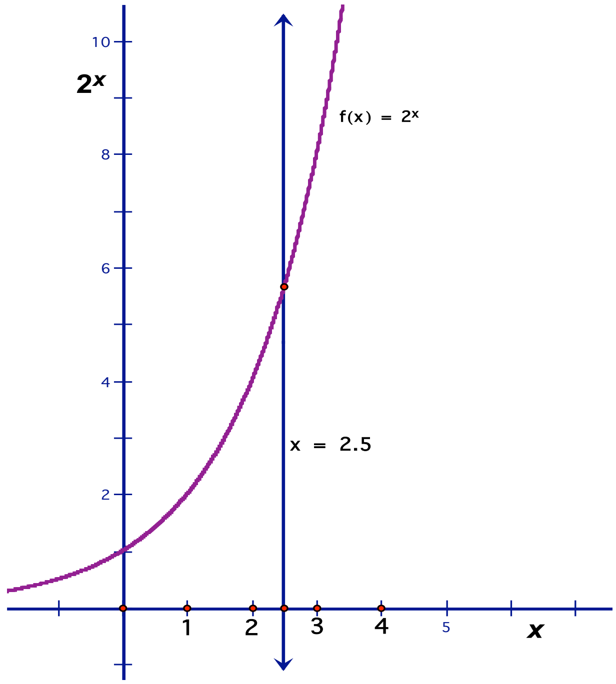
$\left(2^5\right)^{1/2} = \sqrt{2^5} = \sqrt{32} \approx 5.656$ or $(2^{\frac{1}{2}})^5 = (\sqrt{2})^5 \approx 1.414^5 \approx 5.656$.

Mathematical Focus 3

One approach to finding the value of $2^{2.5}$ is to examine the graph of the function $f\left(x\right) = 2^x$.

It should be recognized that at this point one is using the assumption that the plot can be extended from the discrete case of integer exponents to the continuous case. In particular, there were extensions of the domain of the exponent *x* from integers to rational numbers to real numbers, and the graph will be represented by a smooth, continuous function.

One can estimate from the graph the value of the function at $x = 2.5$ in at least two different ways. First, one can look at the intersection of the function graph with the vertical line $x = 2.5$ in the following graph to see $f(x) \approx 5.5$. Second, one can trace along the function graph to obtain $f(x) \approx 5.656$ when $x = 2.5$.



Graphing technologies approximate the solution using invisible algorithms and provide a very accurate image. However, one could plot the points by hand or use a sophisticated drawing tool to construct a smooth curve connecting known values and the value of $2^{2.5}$ could be approximated from that picture.