MAC-CPTM Situations Project

Situation 46: Division Involving Zero

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Prompt

On the first day of class, preservice middle school teachers were asked to evaluate 2/0, 0/0, and 0/2 and to explain their answers. There was some disagreement among their answers for 0/0 and quite a bit of disagreement among their explanations:

- Because any number over 0 is undefined.
- Because you cannot divide by 0.
- Because 0 cannot be in denominator.
- Because 0 divided by anything is 0.
- Because a number divided by itself is 1.

Question: Which of these clauses are explanations?
Question: Which of these clauses apply to 0/0?

Commentary

These responses are not explanations but assertions, and some of them are prohibitions. All of them apply to 0/0, and yet they give three different answers: 0, 1, and undefined.

The explanations below take up 0/0 and 1/0, sometimes beginning by grounding the approach with a description of 12/3. Question: Should the vignette be changed? Or should the explanations be changed by including 0/2 and replacing 1/0 with 2/0.

As you read through the explanations below, pay attention to the important distinction between dividing 0 by 0 and dividing a nonzero number by 0.

Focus 1: Division as the solution to a multiplication problem

If $0/0 = x$, then $0x = 0$. Because any x is a solution to this equation, there is no unique solution. So, 0/0 is undefined.

If $1/0 = x$, then $0x = 0$. No real number x is a solution to this equation, so 1/0 is undefined.

Focus 2: Partitive division

We can think of 12/3 as dividing 12 into 3 groups and asking how many in one group. Another way of saying this is, "If 12 is 3 portions, how many is 1 portion?" This rephrasing helps me to use partitive division to interpret division by non-counting numbers.

In the problem at hand, 0/0 can be restated, "If 0 is 0 portions, how many is 1 portion?" It is not hard to see that there is not enough information to answer the question. If a portion is 3 or 7.2 or any size at all, 0 portions would be 0. Because there is no unique answer, 0/0 is undefined.

Similarly, 1/0 can be restated, "If 1 is 0 portions, how many is 1 portion?" This is impossible! For any portion size, 0 portions cannot be 1 because 0 portions must be 0. So, 1/0 is undefined.

Focus 3: Quotative division

We can think of 12/3 as dividing 12 into groups of 3, and asking how many groups can be made. We can carry out the process by repeated subtraction, removing groups of 3.

For 0/0, we can remove as few or as many groups of 0 as we'd like, and there will be 0 left. In other words, the answer (the number of groups) could be anything. Because there is no unique answer, 0/0 is undefined.

For 1/0, no matter how many groups of 0 we remove, the 1 will remain. Because there is no answer, 1/0 is undefined.

Focus 4: Division as slope

Think of slope, defined as rise/run between two points in the Cartesian plane. Initially, it is important to imagine that the points are distinct, and it is not hard to see that the line will have a unique slope.

But in the case of 0/0, the rise and run are both 0, which means that the two points coincide. Then, geometrically, there is not a unique line through two coincident points. And although any line through the point will have a unique slope, it is impossible to tell which line is the right one. Again, because there is no unique answer, 0/0 is undefined.

In the case of 1/0, the rise is 1 and the run is 0, so the line is vertical. As a line with positive slope becomes vertical, its slope approaches infinity, so we say the slope of a vertical line is undefined.

Focus 5: Division as direct proportion

Suppose $y = kx$ is a direct proportion. For points on the line, the ratio $y/x = k$, which is constant. Because the origin is on the line $y = kx$, it appears that $0/0 = k$. But because this would work for any constant of proportionality, 0/0 could be anything through similar reasoning. So 0/0 is undefined.

[The case of 1/0 is hard to explain via language of direct proportion, but the graph would be a vertical line, and there is a sense in which $k = \infty$, as will be seen below.]

Focus 6: Division as speed

If you go 12 miles in 3 hours, how fast are you going? Answer: 4 miles per hour, and we get the answer through division.
If you go 0 miles in 0 hours, how fast are you going? Any speed works.
If you go 1 mile in 0 hours, how fast are you going? Impossible. [Note that there is a sense of infinite speed here.]

Focus 7: Division as unit price

If $12 buys 3 pounds of tomatoes, how much is 1 pound?
If $0 buys 0 pounds of tomatoes, how much is 1 pound?
If $1 buys 0 pounds of tomatoes, how much is 1 pound?

Focus 8: Division as rate

If Angela makes 3 free throws in 12 attempts, what is her rate?
If Angela makes 0 free throws in 0 attempts, what is her rate?
If Angela makes 1 free throw in 0 attempts, what is her rate?

Note: This focus demonstrates that we need to be a bit careful about giving too much credence to the impossibility of real-world occurrences. Here, I needed to use 3/12 rather than 12/3 as the easy example, because 12/3 would mean 12 free throws in 3 attempts, which is impossible.

Focus 9: Division via rectangle area

Suppose we allow that rectangles can have side lengths of 0.
If a rectangle has area 12 and height 3, what is its width? Answer: 4.
If a rectangle has area 0 and height 0, what is its width? Any width will do.
If a rectangle has area 1 and height 0, what is its width? Impossible. [Note: In fact, this is the Dirac delta function.]

Focus 10: Division via a Cartesian product

There are 12 outfits that can be made using 3 pairs of pants and how many shirts?
There are 0 outfits that can be made using 0 pairs of pants and how many shirts?
There is 1 outfit that can be made using 0 pairs of pants and how many shirts?

Focus 11: Division via factoring

For 12/3, 3 and the quotient are a factor pair for 12. Note that 12 is a multiple of 3.
Regarding 0/0, 0 is part of lots of factor pairs for 0. Note that 0 is a multiple of 0.
For 1/0, 0 is not part of any factor pair for 1. Note that 1 is not a multiple of 0.

Focus 12: As indeterminate forms in calculus

In calculus, 0/0 is an indeterminate form. In the limit, 0/0 can "become" anything. (L'Hospital's rule often helps.) In other words, 0/0 is not a specific number.

In calculus, 1/0 is not usually called an indeterminate form, and usually the limit does not exist. For example, in the case of $\lim_{x\to 0}\frac{1}{x}$, the limit doesn't exist because it approaches $-\infty$ from the left and ∞ from the right. Sometimes it is appropriate to say that the limit is infinity. For example, $\lim_{x\to 0}\frac{1}{x^2}=\infty$.

Side note: Advanced calculus and real analysis texts often distinguish among three cases for the form $a/0$ for $a \neq 0$. (1) The limit is ∞; (2) the limit is $-\infty$; and (3) the limit doesn't exist often because the limits from the left and right are different. We might discuss these ideas using arithmetic of the extended real line, $\Re \cup \{\infty, -\infty\}$, in which ∞ and $-\infty$ are treated like numbers that are the right and left "endpoints" of the real line.

Focus 13: The real projective line

In the Cartesian plane, consider the set of lines through the origin, and consider each line to be an equivalence class of points in the plane.

Except when $x = 0$, the ratio of the coordinates of a point gives the slope of the line that is the equivalence class containing that point. The origin must be excluded because it would be in all equivalence classes, which is rather like saying 0/0 would be the slope of any line through the origin. Note that the slope of a line through the origin is equal to the y-coordinate of the intersection of that line and the line $x = 1$. This way, we can use slope to establish a one-to-one correspondence between the equivalence classes and the real numbers. Thus, the real numbers give us all possible slopes, except for the vertical line.

When $x = 0$, all the points in the equivalence class lie on the vertical line that is the y-axis. (Again the origin must be excluded from this equivalence class.) The ratio of the coordinates is undefined, so the slope is undefined. But this is the only line through the origin that does not have a slope yet, so lets try to give it one. As positively sloped lines approach vertical, their slopes approach ∞, so we might want the slope of the vertical line to be ∞. As negatively sloped lines approach vertical, their slopes approach $-\infty$, which seems to indicate that the slope should instead be $-\infty$. Which should we choose?

Well, there is only one vertical line through the origin, so it cannot have two different slopes. To solve this ambiguity, let's decide that ∞ and $-\infty$ are the same "number" because they should represent the same slope. So now, if we think about all possible slopes, we have all real numbers and one more number, which we will call ∞. Imagine beginning with the extended real line, $\Re \cup \{\infty, -\infty\}$, and gluing together the points ∞ and $-\infty$ so that they are the same point. This is the real projective line, $\Re \cup \{\infty\}$.

Focus 14: Division in general rings

Because we are talking about division involving 0, where division is related to multiplication and 0 is the additive identity, we must be in a system with addition and multiplication, which means we are likely in a ring. (What kind of extended treatments are there of two-operation mathematical structures that are not rings, such as semirings or ringoids?)

In a commutative ring, we can't divide in general, but the expressions $\frac{a}{b}$ and $a \div b$ can make sense if there exists a unique solution to the equation $bx = a$. When b^{-1} exists, both $\frac{a}{b}$ and $a \div b$ can be taken to denote ab^{-1} or $b^{-1}a$, because they are equal.[1] Furthermore, this meaning is consistent with the first meaning, because if $x = b^{-1}a$, then $bx = b(b^{-1}a) = (bb^{-1})a = 1a = a$, as desired. Note that associativity of multiplication is required for these two meanings to be consistent.

To think about $a \div 0$ in a commutative ring, we look for solutions to $0x = a$. The ring axioms imply that $0x = 0$ for all x, which has two consequences:

$0x = a$ has no solutions if $a \neq 0$.
$0x = 0$ is satisfied by any element of the ring.

In a noncommutative ring, the notation $\frac{a}{b}$ is ambiguous because it is unclear, a priori, whether it should denote ab^{-1} or $b^{-1}a$, assuming b^{-1} exists. When b^{-1} does exist, the notation $a \div b$ might readily be interpreted as ab^{-1}, but what can be said when b^{-1} does not exist or is not known to exist? Perhaps $a \div b$ should be a solution to $xb = a$. But in any case, we would also want to consider solutions for $bx = a$, and the reasoning is essentially the same in both cases.

Consider, for example, the matrix equations $BX = A$. If $B = 0$, then A must be zero and X can be any matrix. If we look at the level of determinants (which means that A, X, and B must be square) then $\det(B) = 0$ implies that $\det(A) = 0$ and $\det(X)$ can be anything.

Because most commonly-used two-operation mathematical structures are rings, the previous focus suggests that the question of division by zero is answered pretty completely by the ring axioms. But in many rings there are many other interesting cases of that can be interpreted as dividing 0 by something else. [This sort of discussion is useful for teachers because students often over-generalize "You can't divide by 0" to "You can't divide with 0."]

[1] To be clear, when b^{-1} exists, it is the unique solution to $bx = 1$, where 1 is the multiplicative identity, which means that $bb^{-1} = 1$.

In Z_{12}, 0 ÷ 3 would be a solution to $3x = 0$. There are 3 solutions: $x = 0, 4$, or 8. (Note that 3 is not invertible in Z_{12}.)

In matrix rings, if *B* is a square matrix, $Bx = 0$ has non-zero solutions *x* precisely when the matrix *B* is not invertible. The solutions are called the null space of *B*.

Origins of and audiences for these foci

Only foci 1 and 12 were available to me when I started teaching. Foci 2 and 3 arose from my reading of the mathematics education literature. Focus 4 was suggested by an undergraduate student in my class this past spring. Focus 5 arose this past spring as I considered the direct proportion ideas in the new sixth-grade Georgia Performance Standards. And focus 13 came from my thinking this summer about undergraduate geometry courses for teachers, as I tried to connect ideas of projective and spherical geometry to school mathematics. Perhaps you can see that it is a more formal version of the direct proportion argument.

I have included foci 1, 2, 3, 5, 6, and 8 in a methods course for preservice middle school teachers. Foci 1 through 12 seem appropriate for preservice secondary teachers. Are foci 13 and 14 appropriate for mathematics educators?

Notes

Idea to be developed: Most fallacious proofs involve multiplication or division by 0.

Possible sources of other foci: For other mathematical situations in teaching, I have found explanations in categories such as these:

- Sequences
- Ordering
- Continuity
- As a limit
- Real-world contexts (rates here)
- Linear algebra (matrices or vectors)
- Abstract algebra
- Linearity
- Dimension
- Function
- Common denominator
- Common numerator

Have all of these been exploited for these division situations?

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