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Reflective Discourse and Collective Reflection

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The analysis in this paper focuses on the relationship between classroom discourse and mathematical development. We give particular attention to *reflective discourse*, in which mathematical activity is objectified and becomes an explicit topic of conversation. We differentiate between students' development of particular mathematical concepts and their development of a general orientation to mathematical activity. Specific issues addressed include both the teacher's role and the role of symbolization in supporting reflective shifts in the discourse. We subsequently contrast our analysis of reflective discourse with Vygotskian accounts of learning that also stress the importance of social interaction and semiotic mediation. We then relate the discussion to characterizations of classroom discourse derived from Lakatos' philosophical analysis.

The current reform movement in mathematics education places considerable emphasis on the role that classroom discourse can play in supporting students' conceptual development. The consensus on this point transcends theoretical differences and includes researchers who draw primarily on mathematics as a discipline (Lampert, 1990), on constructivist theory (Cobb, Wood, & Yackel, 1993; Thompson, Philipp, Thompson, & Boyd, 1994), and on sociocultural theory (Forman, 1996; van Oers, 1996). Our purpose in this article is to suggest possible relationships between classroom discourse and the mathematical development of the students who participate in, and contribute to it. To this end, we focus on a particular type of discourse that we call *reflective discourse*. It is characterized by repeated shifts such that what the students and teacher do in action subsequently becomes an explicit object of discussion. In fact, we might have called it *mathematizing discourse* because there is a parallel between its structure and psychological accounts of mathematical development in which actions or processes are transformed into conceptual mathematical objects. In the course of the analysis, we also develop the related construct of *collective reflection*. This latter notion refers to the joint or communal activity of making what was previously done in action an object of reflection.

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In the first part of the article, we discuss action-oriented accounts of mathematical development and then present classroom episodes to exemplify reflective discourse and collective reflection. In subsequent sections, we clarify the students' and teacher's contributions to the development of reflective discourse and consider possible relationships between *individual* students' mathematical development and the classroom *social* processes in which they participate. Against this background, we relate our analysis of classroom discourse to the characterization offered by Lampert (1990), and then conclude by summarizing the pragmatic and theoretical significance of the notion of reflective discourse.

MATHEMATICAL ACTIONS AND MATHEMATICAL OBJECTS

Gray and Tall (1994) recently observed that "the notion of actions or processes becoming conceived as mental objects has featured continually in the literature" (p. 118). As an example, they refer to Piaget's (1972) contention that actions on mathematical entities at one level become mathematical objects themselves at another level. Piaget used the notion of reflective abstraction to account for such developments wherein the result of a mathematical action can be anticipated and taken as a given, and the action itself becomes an entity that can be conceptually manipulated. Mathematics educators working in the Piagetian tradition have called this developmental process "integration" (Steffe, von Glasersfeld, Richards, & Cobb, 1983) and "encapsulation" (Dubinsky, 1991). The influence of this Piagetian view can also be seen in Vergnaud's (1982) discussion of the way in which students gradually explicate theorems-in-action that are initially outside their conscious awareness.

Sfard's (1991) account of mathematical development is compatible with that of Piaget in that she posits a process of reification whereby operational or process conceptions evolve into object-like structural conceptions. She makes an important contribution by grounding this view in detailed historical analyses of a variety of mathematical concepts including number and function. In her view, the historical development of mathematics can be seen as a "long sequence of reifications, each one of them starting where the former ends, each one of them adding a new layer to the complex system of abstract notions" (1991, p. 16). In addition, she illustrates how the development of ways of symbolizing has supported the reification process both historically and in students' conceptual development. Thus, although she is careful to clarify that the development and use of symbols is by itself insufficient, Sfard and Linchevski (1994) contend that mathematical symbols are manipulable in a way that words are not. In her view, this contributes to the reification of activity and to the development of object-like structural conceptions.

The general notion of processes being transformed into objects also features prominently in Freudenthal's analysis of mathematical development. For example, he argues that "the activity of the lower level, that is the organizing activity by means of this level, becomes an object of analysis on the higher level; the operational matter of the lower level becomes a subject matter on the next level" (1973, p. 125). The agreement on this point between Freudenthal and Piaget (1972) is particularly significant given that Freudenthal has criticized other aspects of Piaget's theory. Like Sfard, Freudenthal and his collaborators root much of their work in the history of mathematics. Perhaps because of this commonality, they also emphasize the role that developing and using both informal and standard symbolic schemes can play in supporting the transition from action to object.

It should be noted that this brief discussion of action-oriented developmental theories has necessarily been selective and has omitted a number of important contributions including those of Harel and Kaput (1991), Mason (1989), and Pirie and Kieren (1994). For our purposes, it suffices to note two central assumptions that cut across the work of the theorists we have referenced. The first is that students' sensory-motor and conceptual activity is viewed as the source of their mathematical ways of knowing. The second assumption is that meaningful mathematical activity is characterized by the creation and conceptual manipulation of experientially real mathematical objects. In the remainder of this article, we focus on a type of classroom discourse that appears to support students' reification of their mathematical activity.

BACKGROUND: THE FIRST-GRADE CLASSROOM

The two sample episodes that we will present for illustrative purposes are both taken from a first-grade classroom in which we conducted a year-long teaching experiment. The episodes were selected to clarify the notions of reflective discourse and collective reflection and are not intended to demonstrate exemplary practice. This classroom is of interest in that the mathematical development of all 18 children was significant. Their learning was documented in a series of video-recorded interviews conducted in September, December, January, and May. Our focus in these interviews was on the children's evolving arithmetical conceptions and strategies. A variety of tasks were presented in each set of interviews, including both horizontal number sentences and word problems. The tasks in the September and December interviews involved numbers to 20, whereas those in the January and May interviews involved numbers to 100. In the September interviews, all 18 children used counting strategies that ranged from counting all on their fingers to counting on and counting down. However, only 2 children developed derived-fact or thinking-strategy solutions spontaneously, and each did so on only one occasion. In contrast, 11 children used non-counting strategies routinely in the December interviews to solve all or almost all the tasks presented. Another 5 children used thinking strategies to solve at least half the tasks presented. A comparison of the remaining 2 children's performance in the September and December interviews indicates that they had developed more sophisticated counting methods.

In the January and May interviews, the tasks included word problems such as the following:

> Joe and Bob each grab a handful of candies out of a jar. Joe gets 53 and Bob gets 28. How many does Joe need to put back so that he and Bob have the same number of candies?

The most sophisticated solutions observed in the January interviews occurred when

7 of the children solved addition sentences such as 16 + 9 = ___ and 28 + 13 = ___ by developing efficient noncounting strategies. However, none of the children were able to solve any of the word problems presented. In contrast, 12 of the children solved all or almost all the tasks presented in the May interviews by using relatively efficient noncounting strategies that involved the conceptual coordination of units of ten and one. The remaining 6 children had all made significant progress when compared with their performance in the December and January interviews. Most now used thinking strategies routinely, and all but 1 child attempted to develop solutions that involved units of ten and one.

Detailed analysis of the interviews can be found in Cobb, Gravemeijer, Yackel, McClain, and Whitenack (in press) and Whitenack (1995). Our purpose in giving this very brief summary of the interview results is merely to indicate that, by traditional standards, the first-grade teacher was reasonably successful in supporting her students' mathematical development. Explanations of her success necessarily make reference to a number of issues including the inquiry microculture established in her classroom and the sequences of instructional activities she used. An additional issue that appeared to play an important role concerns the nature of classroom discourse.

REFLECTIVE DISCOURSE: AN INITIAL EXAMPLE

As we have noted, the children used a range of counting methods in the September interviews. The analysis of these interviews also indicated that five of the children could not use their fingers as substitutes for other items. For example, the tasks presented in September included elementary addition story problems such as, "Can you pretend that you have three apples" (child nods affirmatively), "Can you pretend that I have two apples" (child nods), "How many apples do we have together? You have three and I have two." For these five children, the possibility of putting up fingers as perceptual substitutes for the apples did not arise and, as a consequence, they were not able to enter the situation described in the task statement. These findings were consistent with our initial classroom observations of the children and influenced the first sequence of instructional activities that we developed in collaboration with the teacher. This sequence involved finger patterns, spatial patterns, and the conceptual partitioning and recomposing of collections of up to 10 items.

The first episode we will present occurred a week after the September interviews were completed and involved an instructional activity designed to support the development of flexible partitioning (e.g., a collection of five items conceptualized in the imagination as four and one or three and two as the need arises). The teacher used an overhead projector and began by showing a picture of two trees, one larger than the other, and five monkeys. She asked the children several questions about the picture and explained that the monkeys could play in the trees and could swing from one tree to the other. She then asked, "If they all want to play in the trees, just think of ways that we can see all five monkeys in the two trees—all five monkeys to be in two trees."

The collection of monkeys remained visible throughout the ensuing discussion so that all the children might be able to participate by proposing various ways in which the monkeys could be in the trees. The teacher drew a vertical line between the trees and then recorded the children's suggestions, creating a table in the process. For example, the following exchange occurred after the teacher had recorded two children's contributions (shown in Figure 1). The contributions of the five children who could not initially use their fingers as perceptual substitutes are marked with asterisks.

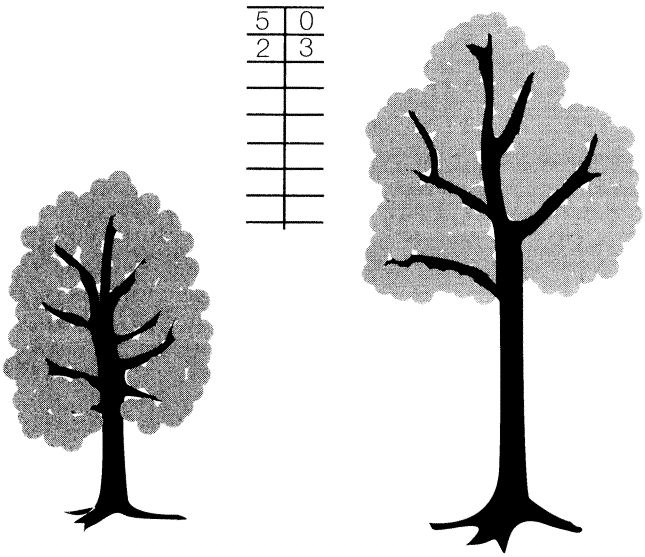

Figure 1. Graphic for the "monkeys in the trees" task

*Anna**: I think that three could be in the little tree and two could be in the big tree.

Teacher: OK, three could be in the little tree, two could be in the big tree [writes 3|2 between the trees]. So, still 3 and 2 but they are in different trees this time; three in the little one and two in the big one. Linda, you have another way?

*Linda**: Five could be in the big one.

Teacher: OK, five could be in the big one [writes 5] and then how many would be in the little one?

*Linda**: Zero.

Teacher: [Writes 0]. Another way? Another way, Jan?

Jan: Four could be in the little tree, one in the big tree.

We note in passing that as part of her role, the teacher gave a commentary from the perspective of one who could judge which aspects of the children's activity might be mathematically significant. Thus, she related Anna's proposal to the previous suggestion by saying, "Still three and two but they are in a different tree this time."

To this point, the theme of the discourse had been generating the possible ways the monkeys could be in the trees. Another child proposed that four (monkeys) could be in the big tree and one in the little tree, resulting in the record of the children's solutions shown in Figure 2. At this point a shift occurred in the discourse:

5	0
2	3
3	2
0	5
4	1
1	4

Figure 2. The record of the children's suggestions

Teacher: Are there more ways? Elizabeth?
*Elizabeth**:* I don't think there are more ways.
Teacher: You don't think so? Why not?
*Elizabeth**:* Because [that's] all the ways that they can be.

Elizabeth presumably made her conjecture because she could not think of another possibility. Another child challenged Elizabeth by proposing an alternative that, it transpired, had already been recorded in the table. In the course of this exchange, the theme became deciding whether there were any more possibilities in what might be termed an empirical way—by generating possibilities and checking them against the table.

A further shift in the discourse occurred when the teacher asked the following question.

Teacher: Is there a way that we could be sure and know that we've gotten all the ways?

Jordan: [Goes to the overhead screen and points to the two trees and the table as he explains.] See, if you had four in this [big] tree and one in this [small] tree in here, and one in this [big] tree and four in this [small] tree, couldn't be that no more. If you had five in this [big] tree and none in this[small] tree, you could do one more. But you've already got it right here [points to 5|0]. And if you get two in this [small] tree and three in that [big] tree, but you can't do that because three in this [small] one and two in that [big] one—there is no more ways, I guess.

Teacher: What Jordan said is that you can look at the numbers and there are only a certain ... there are only certain ways you can make five.

Mark: I know if you already had two up there and then both ways, you cannot do it no more.

Previously, the children had engaged in generating the possible ways the monkeys could be in the trees. Now, the results of that activity were emerging as explicit objects of discourse that could themselves be related to each other. It is this feature of the episode that leads us to classify it as an example of reflective discourse. The significance of this type of discourse lies in its relationship to the accounts of mathematical development given by Piaget (1972), Sfard (1991), and Freudenthal (1973). We conjecture that opportunities arose for the children to reflect on and objectify their prior activity as they participated in the discourse. In other words, the children did not happen to spontaneously begin reflecting at the same moment. Instead, reflection was supported and enabled by participation in the discourse.

To account for this feature of reflective discourse, it might at first glance seem reasonable to suggest that the children engaged in a collective act of reflective abstraction. However, Piaget's (1972) notion of reflective abstraction is a psychological construct that refers to the process by which individual children reorganize their mathematical activity. The contention that the sample episode involved a collective act of reflective abstraction therefore implies that each of the children reorganized their mathematical activity, thereby making a significant conceptual advance. In our view, this assumed direct link between the social process of reflective discourse and individual psychological process of reflective abstraction is too strong. We instead conjecture that children's participation in this type of discourse *constitutes conditions for the possibility of mathematical learning*, but that it does not inevitably result in each child reorganizing his or her mathematical activity (cf. Cobb, Jaworski, & Presmeg, 1996). In this formulation, the link between discourse and psychological processes like reflective abstraction is indirect. This perspective acknowledges that both the process of mathematical learning and its products, increasingly sophisticated mathematical ways of knowing, are social through and through. However, it also emphasizes that children actively construct their mathematical understandings as they participate in classroom social processes. To emphasize this indirect link between the individual and collective aspects of mathematical development, we distinguish between the psychological process of reflective abstraction and the communal activity of *collective reflection* that occurs as children participate in reflective discourse. Thus, in the case of the sample episode, we infer that the children collectively reflected on their prior activity of generating the possible ways the monkeys could be in the two trees. However, inferences about the reflective abstractions and conceptual reorganizations that particular children might have made while participating in the sample episode require a detailed psychological analysis.

This discussion of collective reflection questions the view that the children were merely carried along by a discourse that determined their individual thinking. We further highlight the children's agency by noting that they contributed to the shifts that occurred in the discourse. Consider, for example, the occasion when the teacher asked the children if there was a way that they could be sure they had generated all the possibilities. It was at this juncture that Jordan paired up possibilities involving the same number combinations. In giving this explanation, he reflected on and reorganized the class's prior activity as recorded in the table. Had none of the children been able to respond to the teacher's question in this way, the shift in discourse would not have occurred. It is therefore reasonable to say that individual children contributed to the development of the discourse that supported and sustained collective reflection.

REFLECTIVE DISCOURSE AND MATHEMATICAL LEARNING

We have taken care to clarify that the proposed relationship between reflective discourse and conceptual development in mathematics is speculative. In considering what children might learn when they engage in such discourse, we distinguish between (a) their construction of specific mathematical conceptions and (b) the general orientation to mathematical activity that participation in the discourse might foster.

Conceptual Development in Discourse

With regard to the first of these two issues, there is some indication that the discussions of partitioning were productive for some of the children. The partitioning activity was repeated in the monkeys-and-trees setting the day after the sample episode, and then again 3 weeks later using the setting of a double-decker bus (van den Brink, 1989). The children were told that there were, say, ten passengers on a double decker bus and were asked to decide how many could be on the top deck and how many could be on the bottom deck. On both days, the teacher recorded the children's suggestions, and the possibility of organizing their proposals into pairs became an explicit topic of conversation. On the second day, the children were also asked to complete an activity sheet that involved generating partitions for the the double-decker bus problem. Of the 16 children present, 8 consistently generated commutative pairs and 3 produced more sophisticated organizations. For a task involving 7 passengers, the possibilities included

0	7	6	5	4	3	2	1
7	0	1	2	3	4	5	6

and

7	0	6	1	5	2	4	3
0	7	1	6	2	5	3	4

where

7
0

signified 7 people on the top deck of the bus and none on the bottom deck. No discernible patterns could be detected in the written work of the remaining 5 children.

Significantly, 4 of these latter children were among the 5 who had been unable to use their fingers as perceptual substitutes in the September interviews. It therefore seems important to consider how these children might have interpreted the prior whole-class discussions. Possibly, they did not reorganize the results of prior activity when they attempted to understand what other children and the teacher were saying and doing (cf. Steffe, Firth, & Cobb, 1981). For them, the discourse might have been about finding ways the monkeys could be in the trees or the passengers could be on the bus, per se. It should be noted that this possibility would constitute an advance when compared with their activity in the September interviews.

The analysis of the children's written work serves to emphasize further that although participation in reflective discourse supports and enables individual reflection on, and reorganization of, prior activity, it does not cause it, determine it, or generate it. Thus, in the view we are advancing, it is the individual child who has to do the reflecting and reorganizing while participating in and contributing to the development of the discourse. This implies that the discourse and the associated communal activity of collective reflection both support and are constituted by the constructive activities of individual children.

The Development of a Mathematizing Orientation

We can address the second aspect of the students' learning—their general orientation to mathematical activity—by considering an episode that occurred in the same classroom almost 5 months after the first. The instructional activities used at the time of the second sample episode were designed to support the children's structuring of numbers up to 100 into composite units, particularly of ten and one. One of the anchoring situations that the teacher had previously introduced was that of Mrs. Wright's candy shop, where loose candies were packed into rolls of ten. To introduce this particular instructional activity, the teacher explained that Mrs. Wright was interrupted when she was counting out candies and putting them into rolls. She then posed the following question:

Teacher: What if Mrs. Wright has 43 pieces of candy and she is packing them into rolls. What are different ways that she might have 43 pieces of candy—how many rolls and how many pieces might she have? Sarah, what's one way she might find them?

The discussion proceeded smoothly, in that the children, as a group, generated the various possibilities with little apparent difficulty. Their contributions were as follows:

Sarah:* Four rolls and three pieces.

Elizabeth:* Forty-three pieces.

Kendra: She might have two rolls and 23 pieces.

Darren: She could have three rolls, 12 pieces, I mean 13 pieces.

Linda:* One roll and 33 pieces.

The teacher for her part recorded each of their suggestions on a board as shown in Figure 3.

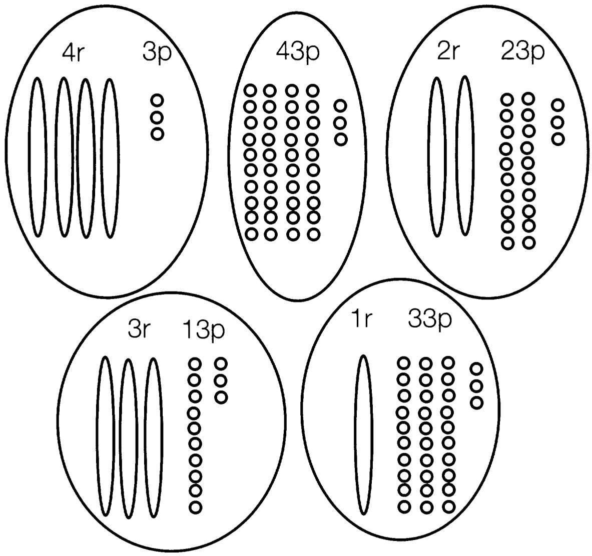

Figure 3. The teacher's record of the students' suggestions

Karen then indicated that she had something she wanted to say and went to the board at the front of the room.

Karen: Well, see, we've done all the ways. We had 43 pieces....

Teacher: OK.

Karen: And, see, we had 43 pieces [points to 43p] and right here we have none rolls, and right here we have one roll [points to 1r 33p]

Teacher: OK, I'm going to number these, there's one way ... no rolls [writes "0" next to 43p].

Karen: And there's one roll, there's 2 rolls, then there's 3, and there's 4.

Teacher: [Numbers the corresponding pictures 1, 2, 3, 4].

This exchange is reminiscent of the first episode in that both concern the possible ways of partitioning a specified collection. However, a crucial difference between the two episodes concerns the justifications given for the claim that all possibilities had been found. In the first episode, Elizabeth explained that she could not think of any more ways. It was not until the teacher asked the children how they could know for sure that they had found all the ways that the discourse moved beyond what might be termed empirical arguments. In contrast, Karen justified her claim by ordering the possibilities that had been generated without prompting. Thus, almost seamlessly, the discourse shifted from generating the possible ways the candies might be packed to operating on the results of that generative activity.

As the exchange continued, it became apparent that many other children took the need to produce an ordering as self-evident. Jan joined Karen at the board and proposed an alternative way of numbering the pictures:

Jan: I think you should number them, like put them in order. Like that one first [points to 43p], then that one [1r 33p].

Teacher: [Erases the 0 next to 43p and writes 1] Call this number two [1r 33p]?

Jan: 'Cause that's the first way [43p], 'cause it's no rolls.

Teacher: OK. [Writes 2 and 3 next to 1r 33p and 2r 23p, respectively.]

However, at this point, Jan began to focus on the number of rolls in each configuration rather than on the order that each configuration would be produced when packing candy and eventually said that she was confused.

Another child, Casey, said that he thought he knew what Jan meant and joined her and Karen at the board.

Casey: She means like there's none right here [43p] and that's [number] one, and then there's one [roll] right here [1r 33p], that makes [number] 2; there's two [rolls] right here [2r 23p], that makes [number] 3; there's three [rolls] right here [3r 13p], that makes [number] four; and there's four [rolls] right there [4r 3p]; that makes [number] five.

Teacher: [Labels the configurations as Casey speaks.]

Casey: Because that's the first one [43p], that's the second [1r 33p] one, that's the third [2r 23p]....

Jan: No, I think that should be the third [3r 13p].

Casey: I'm not counting rolls.

As the exchange continued, it became increasingly evident that Jan, Karen, and a third child, Anna*, on one side, and Casey on the other side were talking past each other. Jan, Karen, and Anna* repeated that their numbering scheme was based on the number of rolls, and Casey continued to protest that he was not counting rolls. At this juncture, the teacher intervened and initiated a final shift in the discourse. She first explained Jan, Karen, and Anna's approach, stressing that "the way that they were thinking about numbering them and naming them was by how many rolls that they used to make 43 candies."

Casey: I wasn't counting the rolls, I was counting how they went in order. Like that one was the first one, and that one was the second one.

[Several children indicated they disagreed or did not understand.]

Casey: I wasn't counting the rolls. I wasn't going like "one roll, two rolls, three rolls, four rolls."

Teacher: OK, Casey, now let me say what you were saying and you listen to me and see if I say what you said. [Addresses the class.] Now what it means is that there are two different ways of naming them.

She then went on to contrast the rationales for the two approaches, stressing that "Casey was talking about just naming them a different way." Jan interrupted the teacher to say, "Now I understand," and then spontaneously explained Casey's approach herself.

Two major shifts in the discourse can be discerned in this second episode. In the first, the various partitionings of 43 candies that the children generated were objectified and treated as entities that could be ordered. In the second shift, the activity of ordering the configurations itself became an explicit topic of conversation. To the extent that the children participated in these shifts, they could be said to be engaging in the activity of mathematizing. The first of these shifts in which the partitionings were treated as entities was relatively routine for many of the children. This suggests that the general orientation the children were developing when they participated in reflective discourse was that of mathematizing. In this regard, we note with Bauersfeld (1995):

> Participating in the processes of a mathematics classroom is participating in a culture of mathematizing. The many skills, which an observer can identify and will take as the main performance of the culture, form the procedural surface only. These are the bricks of the building, but the design of the house of mathematizing is processed at another level. As it is with culture, the core of what is learned through participation is *when* to do what and *how* to do it .... The core part of school mathematics enculturation comes into effect on the meta-level and is "learned" indirectly. (p. 460)

Our conjecture is that a crucial aspect of what is currently called a mathematical disposition (National Council of Teachers of Mathematics, 1991) is developed in this indirect manner as children participate in reflective classroom discourse.

SYMBOLIZING AND THE TEACHER'S ROLE

Thus far, we have considered how participation in reflective discourse might support students' mathematical learning. In doing so, we have conjectured about both specific conceptual developments that involve the transition from process to object and the more general development of a mathematizing orientation. The notion of reflective discourse also helps clarify certain aspects of the teacher's role. In our view, one of the primary ways in which teachers can proactively support students' mathematical development is to guide and, as necessary, initiate shifts in the discourse such that what was previously done in action can become an explicit topic of conversation. This was exemplified in the first episode when the teacher initiated a shift beyond what we termed empirical verification by asking, "Is there a way that we could be sure and know that we've gotten all the ways [that five monkeys could be in the two trees]?" The ensuing shift in the discourse that occurred can be viewed as an interactional accomplishment in that it also depended on the contribution made by one of the children, Jordan. The role that the teacher's question played in this exchange was, in effect, that of an invitation, or an offer, to step back and reorganize what had been done thus far.[1]

It is important to clarify that initiating and guiding the development of reflective discourse requires considerable wisdom and judgment on the teacher's part. One can, for example, imagine a scenario in which a teacher persists in attempting to initiate a shift in the discourse when none of the students gives a response that involves reflection on prior activity. The very real danger is, of course, that an intended occasion for reflective discourse will degenerate into a social guessing game in which students try to infer what the teacher wants them to say and do (cf. Bauersfeld, 1980; Voigt, 1985). In light of this possibility, the teacher's role in initiating shifts in the discourse

[1]It can be argued that the instructional activities should be modified so that the need to "know for sure" arises in an apparently spontaneous way. However, as a practical matter, this ideal is not always obtainable. It is therefore essential that the teacher be prepared to take the initiative in facilitating shifts in the discourse.

might be thought of as probing to assess whether children can participate in the objectification of what they are currently doing. Such a formulation acknowledges the teacher's proactive role in guiding the development of reflective discourse while simultaneously stressing both that such discourse is an interactional accomplishment and that students necessarily have to make an active contribution to its development.

A second aspect of the teacher's role apparent in both episodes is the way in which she developed symbolic records of the children's contributions. Of course, one can imagine a scenario in which ways of notating could themselves have been a topic for explicit negotiation. For our purpose, the crucial point is not who initiated the development of the notational schemes, but the fact that the records grew out of students' activity in a bottom-up manner (cf. Gravemeijer, in press), and that they appeared to play an important role in facilitating collective reflection on that prior activity. We can clarify both the influence of the records and their supportive role by considering the first of the two episodes. There, the influence of the table the teacher made while recording the children's contributions first became apparent during the following exchange:

Linda:* Five [monkeys] could be in the big one.

Teacher: OK, five could be in the big one [writes 5] and then how many would be in the little one?

Linda:* Zero.

Teacher: [Writes 0.]

Here, the teacher presumably asked Linda how many monkeys were in the small tree only because she was recording Linda's proposal in the table. In other words, the table influenced what counted as a complete contribution. Later in the episode, the table greatly facilitated the process of empirically checking whether particular partitionings of the five monkeys had already been proposed. This was the first occasion when entries in the table were pointed to, and spoken of, as signs that signified various partitionings. Thus, there was a reversal of the signifier-signified relation (cf. Kaput, 1991). Later, when Jordan explained his approach of pairing the partitionings, he pointed to the table entries as he spoke about monkeys in trees. In doing so, he appeared to *look through* the table entries to see the partitionings about which he spoke. Thus, although we as observers can distinguish between the signified and signifier, the table entries and the various partitionings, the two seemed to be inseparable for him in that the table entries *meant* particular partitionings of monkeys.

We could give a similar account of the role that the pictures of candies played in the second episode. There again, the distinction between signified and signifier appeared to be reversed in the course of the episode. Further, as in the first episode, shifts in the discourse were accompanied by changes in the function of the symbolic records until, eventually, the records became explicit objects of discourse that the children looked through to see the prior activity that the records symbolized. Both here and in the first episode, the important role attributed to symbolization is highly compatible with the theory of realistic mathematics education developed by Freudenthal and his collaborators (cf. Treffers, 1987; Streefland, 1991). It is also consistent with the results of Sfard's (1991) historical analyses, which indicate that the development of ways of notating has repeatedly played a central role in the mathematician's transition from processlike operative conceptions to objectlike structural conceptions. This is not to say, however, that either the development of ways of symbolizing or changes in discourse inevitably lead to changes in individual students' thinking. We further clarify our position on this issue in the next section.

THE INDIVIDUAL IN DISCOURSE

There are strong parallels between our discussion of reflective discourse and Vygotsky's (1978) sociohistorical analysis of development. The reader might therefore assume that the analysis we have presented is a straightforward elaboration of Vygotsky's position. To guard against this interpretation, we take Vygotsky's work as a point of contrast. Although we will emphasize differences in perspectives, the intellectual debt we owe Vygotsky should be readily apparent. We have, in fact, refined our position by engaging in an argument with the voice of Vygotsky as expressed in his writings.

Vygotsky emphasized two primary influences on conceptual development, social interaction and semiotic mediation (cf. van der Veer & Valsiner, 1991). With regard to the first of these influences, he posited that interpersonal relations are internalized from the interpsychological plane and reconstituted to form the intrapsychological plane of psychological functions. Our speculation that the first graders developed a mathematizing orientation as they participated in reflective discourse might appear to exemplify this process of internalization from the interpersonal domain to the psychological domain. The reader could therefore conclude that we have followed Vygotsky in contending that the children internalized distinctive aspects of this collective activity.

In our discussion of reflective discourse, we also addressed the second influence on conceptual development considered by Vygotsky, that of semiotic mediation. Here again, Vygotsky gave social and cultural processes priority over individual psychological processes. He argued that in the course of development, cultural tools such as oral and written language are internalized and become psychological tools for thinking (cf. Rogoff, 1990). In the two episodes, the means of notating the teacher used to record the children's contributions could be characterized as cultural tools, and the changes that occurred in the role played by the symbols could be interpreted as steps in this internalization process. One might, in fact, be tempted to follow Leont'ev (1981) and talk of the students appropriating the cultural tools introduced by the teacher. In such an account, the ways of notating would be treated as objective mediators that served to carry mathematical meaning from one generation to the next.

Before differentiating our position from that of Vygotsky, we must stress that we do agree with many of the central tenets of his theory. For example, we agree that children's mathematical development is profoundly influenced both by the face-to-face interactions and by the cultural practices in which they participate. In addition, we believe that Vygotsky was right when he contended that thinking is inextricably bound up with the cultural tools that are used (cf. Dörfler, 1996;

Kaput, 1991; Thompson, 1992). Thus, as Lesh and Lamon (1992) note, it is difficult for us to imagine how the world might have been experienced before the conceptual models and associated notation schemes that we now take for granted were developed. The issue at hand is not, therefore, that of determining whether social and cultural processes influence individual psychological processes. Instead, it concerns the specific nature of the relationship between these two domains, and it is here that we depart from Vygotskian theory.

Vygotsky (1978) argued that the qualities of mental development are *derived from*, and *generated by*, the distinctive properties of the sociocultural organization of the activities in which the individual participates (van Oers, 1996). The linkage he proposed between the two domains is therefore a *direct* one (cf. Cobb, Jaworski, & Presmeg, 1996). This, it will be recalled, is the relation that we questioned when we introduced the notion of collective reflection. The acceptance of this relation implies that children's development of a mathematizing attitude can be accounted for directly in terms of their participation in reflective discourse, without the need to refer to their individual constructive activities. Elsewhere, we have noted that this approach emphasizes the social and cultural basis of personal experience (Cobb, 1995). In our view, it is an appropriate approach to take when addressing a variety of issues including those that pertain to diversity and the restructuring of the school. Therefore, our intent is not to reject this approach out of hand. We do question, however, the explanatory power it provides in addressing the issues under discussion in this article. What is required is an analytical approach that is fine-grained enough to account for qualitative differences in individual children's thinking even as they participate in the same collective activities (cf. Confrey, 1995). The relevance of this criterion became apparent during the discussion of the first episode when we observed that 5 of the 16 children did not appear to have reorganized prior partitioning activity. This indicates that the thinking of some children but not others reflected the organization of the social activity in which they participated. Our rationale for positing an indirect linkage between social and psychological processes is therefore pragmatic and derives from our desire to account for such differences in individual children's activity. As we have noted, this view implies that participation in an activity such as reflective discourse constitutes the conditions for the possibility of learning, but it is the students who actually do the learning. Participation in reflective discourse, therefore, can be seen both to enable and constrain mathematical development, but not to determine it.

Given the current interest in the philosophy and pedagogy of John Dewey (1926), we close this discussion of the relation between individual and social processes by noting that the position we have outlined is closer to that of Dewey than to Vygotsky (1978). For Dewey, as for Vygotsky, learning was a process of enculturation into growing and changing traditions of practice. However, Dewey also stressed the contributions of actively interpreting students. Similarly, we have attempted to illustrate that students contribute to the development of the communal activities in which they participate. The following summary that Dewey gave of his position has a remarkably contemporary ring to it:

> The customs, methods and *working* standards of the calling constitute a "tradition," and initiation into the tradition is the means by which the powers of learners are released and directed. But we should also have to say that the urge or need of an individual to join in an undertaking is a necessary prerequisite of the tradition's being a factor in his personal growth in power and freedom; and also that he has to *see* on his own behalf and in his own way the relations between means and methods employed and results achieved. Nobody else can see for him and he can't see by just being "told," although the right kind of telling may guide his seeing and thus help him see what he needs to see. (1926, p. 57; italics in original. Quoted by Westbrook, 1991, p. 505)

DISCOURSE ON DISCOURSE

Thus far, we have placed our analysis of reflective discourse in a broader theoretical context by contrasting it with Vygotskian theory. We now relate it to Lampert's (1990) influential discussion of discourse in reform classrooms. Lampert derives her vision of the ideal form of classroom discourse from a consideration of mathematics as a discipline. Drawing particularly on Lakatos (1976) and Pólya (1954), she argues that the discourse of mathematicians is characterized by a zig-zag from conjectures to an examination of premises through the use of counter-examples. One of her primary goals is to investigate whether it is possible for students to engage in mathematical activity congruent with this portrayal of disciplinary discourse. As a consequence, her focus is, for the most part, on how the teacher and students interact as they talk about and do mathematics. We will suggest that our discussion of reflective discourse complements this work by focusing on what the teacher and students create individually and collectively in the course of such interactions. First, however, we tease out further aspects of Lampert's analysis.

We note with Billig (1987) that a zig-zag between the general and the particular, or between conjectures and refutations, is not specific to mathematics, but is characteristic of argumentation in general. A further aspect of Lampert's (1990) analysis differentiates mathematical and scientific discourse from other kinds of discourse. She gives particular emphasis to three maxims that Pólya (1954) believed are essential when making "a ready ascent from observations to generalizations, and a ready descent from the highest generalizations to the most concrete observations" (p. 7). These maxims concern the *intellectual courage* and *intellectual honesty* needed to revise a belief when there is a good reason to change it, and the wise restraint that should be exercised so that beliefs are not changed wantonly and without good reason. Significantly, these maxims and the related vision of classroom discourse appear to correspond closely to four commitments that Bereiter (1992) contends are central to scientific discourse and make scientific progress possible:

1. A commitment to work toward common understanding satisfactory to all;
2. A commitment to frame questions and propositions in ways that enable evidence to be brought to bear on them;
3. A commitment to expand the body of collectively valid propositions; and

4. A commitment to allow any belief to be subjected to criticism if it will advance the discourse. (p. 8)

Bereiter contends that these four commitments distinguish mathematical and scientific discourse from other forms of discourse, including philosophical, legal, and political discourse. His analysis therefore substantiates Lampert's claim that she is teaching her students to "act on the basis of what Pólya calls 'the moral qualities of the scientist'" (1990, p. 58). In this respect, her analysis makes an important contribution.

In considering the substance of classroom discourse, we observe with Gravemeijer (in press) that Lampert (1990) takes pure mathematics as her model when she draws on Lakatos (1976) and Pólya (1954). Gravemeijer argues that applied mathematics can also be an important source of analogies when attention centers on students' mathematical development, particularly at the elementary school level. In developing this analogy, he suggests that

> part of a [classroom] discussion is about the interpretation of the situation sketched in the [applied] contextual problem. Another part of the discussion focuses on the adequacy and the efficiency of various solution procedures. This can implicate a *shift* of attention towards a *reflection* on the solution procedure from a mathematical point of view. [Emphasis added.]

This sketch is highly compatible with our account of reflective discourse. The two sample episodes both focus on what Gravemeijer would call context problems and involve shifts that lead to the development of collective reflection. Consequently, whereas Lampert's primary concern is with the commitments that make progress possible as arguments zig-zag from conjectures to refutations, we are more interested in the process of mathematization as it occurs in the course of such discussions. It is in this sense that we suggest the two approaches are complementary.

CONCLUSIONS

Throughout the discussion, we have suggested that the notion of reflective discourse is of interest for both pragmatic and theoretical reasons. Pragmatically, an analysis of reflective discourse clarifies how teachers might proactively support their students' mathematical development in ways compatible with recent reform recommendations. It therefore has implications for in- and preservice teacher development. In this regard, we have used the notion of reflective discourse to guide the editing of classroom videorecordings when preparing hypermedia cases for teacher development (Goldman et al., 1994). This notion also proved useful when developing instructional activities both in collaboration with the first-grade teacher and at other classroom research sites. In particular, instructional sequences were frequently designed so that it might be possible for the teacher to initiate shifts in the discourse by capitalizing on the students' mathematical contributions.

Theoretically, we have argued that reflective discourse is a useful construct in that it suggests possible relationships between classroom discourse and mathematical development. This issue is of considerable significance given the emphasis placed on discourse in current reform recommendations. It is, however, important to acknowledge that reflective discourse is a sociological construct. The delineation of shifts in classroom discourse traces developments in the practices established by the classroom community. Therefore, analyses are needed that systematically coordinate accounts of such communal developments with detailed analyses of individual students' mathematical activity as they participate in, and contribute to, shifts in the discourse. This, in our view, is a productive avenue for further investigation.

REFERENCES

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. *Educational Studies in Mathematics, 11*, 23–41.

Bauersfeld, H. (1995). "Language games" in the mathematics classroom—their function and the education of teachers. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 271–291). Hillsdale, NJ: Erlbaum.

Bereiter, C. (1992, April). *Implications of postmodernism for science education: Science as progressive discourse*. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Billig, M. (1987). *Arguing and thinking: A rhetorical approach to social psychology*. New York: Cambridge University Press.

Cobb, P. (1995). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23*(7), 13–21.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, L., Whitenack, J. (in press). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirschner & J. A. Whitson (Eds.), *Situated cognition theory: Social, semiotic, and neurological perspectives*. Hillsdale, NJ: Erlbaum.

Cobb, P., Jaworski, B., & Presmeg, N. (1996). Emergent and sociocultural perspectives on mathematical activity. In P. Nesher, L. Steffe, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning*. Hillsdale, NJ: Erlbaum.

Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In N. Minick, E. Forman, & A. Stone (Eds.), *Education and mind: Institutional, social, and developmental processes*. Oxford, England: Oxford University Press.

Confrey, J. (1995). How compatible are radical constructivism, sociocultural approaches, and social constructivism? In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp.185–225). Hillsdale, NJ: Erlbaum.

Dewey, J. (1926/1981–1989). Individuality and experience. In J. A. Boydston (Ed.), *John Dewey: The later works, 1925–1953*. Carbondale, IL: Southern Illinois University Press.

Dörfler, W. (1996). Computer use and views of the mind. In C. Keitel & K. Ruthven (Eds.), *Learning from computers: Mathematics education and technology*. New York: Springer Verlag.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–123). Dordrecht, The Netherlands: Kluwer.

Forman, E. (1996). Forms of participation in classroom practice: Implications for learning mathematics. In P. Nesher, L. Steffe, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 115–130). Hillsdale, NJ: Erlbaum.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, The Netherlands: Reidel.

Goldman, E., Barron, L., Bassler, O., Cobb, P., Bowers, J., McClain, K., Robinson, C., St. Clair, J., Harwood, J., & Altman, J. (1994). Investigations in teaching geometry [CD-ROM and computer program]. Nashville, TN. (Provided in association with National Science Foundation Grants No. TPE-8751472, TPE-8950310, TPE-9053826, and TPE-9154067.)

Gravemeijer, K. (in press). Mediating between concrete and abstract. In T. Nunes & P. Bryant (Eds.), *How do children learn mathematics?* Hillsdale, NJ: Erlbaum.

Gray, E. M., & Tall, D. D. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*, 116–146.

Harel, G., & Kaput, J. J. (1991). The role of conceptual entities in building advanced mathematical concepts and their symbols. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 82–94). Dordrecht, The Netherlands: Kluwer.

Kaput, J. J. (1991). Notations and representations as mediators of constructive processes. In E. von Glasersfeld (Ed.), *Constructivism in mathematics education* (pp. 53–74). Dordrecht, The Netherlands: Kluwer.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge, MA: Cambridge University Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29–63.

Leont'ev, A. N. (1981). The problem of activity in psychology. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology*. Armonk, NY: Sharpe.

Lesh, R., & Lamon, S. (1992). Assessing authentic mathematical performance. In R. Lesh & S. J. Lamon (Eds.), *Assessment of authentic performance in school mathematics* (pp. 17–62). Washington, DC: American Association for the Advancement of Science.

Mason, J. (1989). Mathematical abstraction as the result of a delicate shift of attention. *For the Learning of Mathematics, 9*(2), 2–8.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Piaget, J. (1972). *The principles of genetic epistemology*. London: Routledge & Kegan Paul.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*, 61–86.

Pólya, G. (1954). *Induction and analogy in mathematics*. Princeton, NJ: Princeton University Press.

Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. Oxford: Oxford University Press.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1–36.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification—the case of algebra. *Educational Studies in Mathematics, 26*, 87–124.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. New York: Praeger Scientific.

Steffe, L. P., Firth, D., & Cobb, P. (1981). On the nature of counting activity: Perceptual unit items. *For the Learning of Mathematics, 1*(2), 13–21.

Streefland, L. (1991). *Fractions in realistic mathematics education. A paradigm of developmental research*. Dordrecht, The Netherlands: Kluwer.

Thompson, P. W. (1992). Notations, principles, and constraints: Contributions to the effective use of concrete manipulatives in elementary mathematics. *Journal for Research in Mathematics Education, 23*, 123–147.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In D. B. Aichele (Ed.), *Professional Development for Teachers of Mathematics*, The 1994 Yearbook of the National Council of Teachers of Mathematics (pp. 79–92). Reston, VA: National Council of Teachers of Mathematics.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction—the Wiskobas Project*. Dordrecht, The Netherlands: Reidel.

van den Brink, F. J. (1989). *Realistisch rekenonderwijs aan jonge kinderer*. Utrecht, The Netherlands: Vakgroep Onderzoek Wiskundeonderwijs & Onderwijscomputer Centrum, Rijksuniversiteit Utrecht.

van der Veer, R., & Valsiner, J. (1991). *Understanding Vygotsky: A quest for synthesis*. Cambridge, MA: Blackwell.

van Oers, B. (1996). Learning mathematics as meaningful activity. In P. Nesher, L. Steffe, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 91–114). Hillsdale, NJ: Erlbaum.

Vergnaud, G. (1982). Cognitive and developmental psychology and research in mathematics education: Some theoretical and methodological issues. *For the Learning of Mathematics, 3*(2), 31–41.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactique des Mathematiques, 6*, 69–118.

Vygotsky, L. S. (1978). *Mind and society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Westbrook, R. B. (1991). *John Dewey and American democracy*. Ithaca, NY: Cornell University Press.

Whitenack, J. W. (1995). *Modeling, mathematizing, and mathematical learning as it is situated in the classroom microculture*. Unpublished doctoral dissertation, Vanderbilt University.

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