MAT 195 – Fall Quarter 2002
TEST 4 – Answers

NAME____

Show work and write clearly.

For #1 - 5, find the derivative. Simplify all answers.

1. (6 pts.) $y = e^{\sin(5x)}$ **ANS:** (use chain rule): $y' = 5\cos(5x)e^{\sin(5x)}$

2. (6 pts.) $y = \dfrac{x}{\sqrt{7-3x}}$ **ANS:** (use quotient and chain rules):

$$y' = \frac{\sqrt{7-3x}(1) - x\frac{1}{2}(7-3x)^{-1/2}(-3)}{\left(\sqrt{7-3x}\right)^2} = \frac{\sqrt{7-3x} + \frac{3}{2}x(7-3x)^{-1/2}}{7-3x}$$

3. (6 pts.) $y^5 + x^2y^3 = 1 + ye^{x^2}$ **ANS:** (use implicit differentiation and solve for y'. use product rule for x^2y^3 and product and chain rules for ye^{x^2}):

$\Rightarrow 5y^4 \cdot y' + 2x \cdot y^3 + x^2 \cdot 3y^2 \cdot y' = 0 + y'e^{x^2} + y \cdot e^{x^2} \cdot 2x$

Rearrange so that all terms with y' are on one side of equal sign:

$\Rightarrow -y'e^{x^2} + 5y^4 \cdot y' + x^2 \cdot 3y^2 \cdot y' = y \cdot e^{x^2} \cdot 2x - 2x \cdot y^3$

Factor out y' and solve for y':

$$\Rightarrow y'\left(-e^{x^2} + 5y^4 + 3x^2y^2\right) = 2xye^{x^2} - 2xy^3 \Rightarrow y' = \frac{2xye^{x^2} - 2xy^3}{-e^{x^2} + 5y^4 + 3x^2y^2}$$

4. (6 pts.) $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$ **ANS:** (use implicit differentiation and solve for y'):

$\Rightarrow 6y^2 \cdot y' + 2y \cdot y' - 5y^4 \cdot y' = 4x^3 - 6x^2 + 2x$

$$\Rightarrow y'\left(6y^2 + 2y - 5y^4\right) = 4x^3 - 6x^2 + 2x \Rightarrow y' = \frac{4x^3 - 6x^2 + 2x}{6y^2 + 2y - 5y^4}$$

5. (7 pts.) $y = x^{\csc(2x)}$ **ANS:** (use logarithmic differentiation):

Take ln of both sides

$\Rightarrow \ln y = \ln x^{\csc(2x)}$

Use power property of logs:

$\Rightarrow \ln y = \csc(2x)\ln x$

Differentiate implicitly, using chain and product rules for right side:

$$\Rightarrow \frac{1}{y} \cdot y' = -2\csc(2x)\cot(2x) \cdot \ln x + \csc(2x) \cdot \frac{1}{x}$$

Solve for y':

$$\Rightarrow y' = \left(-2\csc(2x)\cot(2x) \cdot \ln x + \csc(2x) \cdot \frac{1}{x}\right)y = \left(-2\csc(2x)\cot(2x) \cdot \ln x + \csc(2x) \cdot \frac{1}{x}\right)x^{\csc(2x)}$$

6. (7 pts.) $y = \sec^{-1}\left(e^x\right)$ **ANS:** (use chain rule): $y' = \dfrac{1}{e^x\sqrt{\left(e^x\right)^2 - 1}} \cdot \left(e^x\right) = \dfrac{1}{\sqrt{e^{2x} - 1}}$

For #7-9, find the second derivative. Simplify all answers.

7. (7 pts.) $y = \sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x^2}}$ **ANS:** Rewrite as $y = x^{1/3} + x^{-2/3}$. So, $y' = \dfrac{1}{3}x^{-2/3} - \dfrac{2}{3}x^{-5/3}$ and

$$y'' = -\frac{2}{3}\frac{1}{3}x^{-5/3} - \left(-\frac{5}{3}\right)\frac{2}{3}x^{-8/3} = -\frac{2}{9}x^{-5/3} + \frac{10}{9}x^{-8/3}$$

8. (7 pts.) $x^6 + y^6 = 1$ **ANS:** Rewrite as $y = \sqrt[6]{1 - x^6}$ and use chain rule to find derivatives. **<u>OR</u>** differentiate implicitly: $6x^5 + 6y^5 \cdot y' = 0 \Rightarrow y' = \dfrac{-6x^5}{6y^5} = \dfrac{-x^5}{y^5}$. To find second derivative use chain and quotient rules: $y'' = \dfrac{y^5 \cdot -5x^4 - \left(-x^5 \cdot 5y^4 \cdot y'\right)}{\left(y^5\right)^2} = \dfrac{-5y^5x^4 + 5x^5y^4 \cdot y'}{y^{10}}$. This simplifies to $y'' = \dfrac{y^4\left(-5yx^4 + 5x^5 \cdot y'\right)}{y^{10}} = \dfrac{-5yx^4 + 5x^5 \cdot y'}{y^6}$. Or you can substitute for y′:

$$y'' = \frac{-5y^5x^4 + 5x^5y^4 \cdot \dfrac{-x^5}{y^5}}{y^{10}} = \frac{-5y^6x^4 + 5x^{10}}{y^{11}}$$

9. (7 pts.) $y = \ln(\cos(x))$ **ANS:** (Use chain rule): $y' = \dfrac{1}{\cos x} \cdot -\sin x = -\tan x$. So, $y'' = -\sec^2 x$.

10. (6 pts.) Find the absolute maximum and the absolute minimum values of *f* on the given interval: $f(x) = 10 + 27x - x^3$, [0, 4]. **ANS:** First, find critical numbers by setting derivative equal to zero: $f'(x) = 27 - 3x^2 \overset{set}{=} 0 \Rightarrow 27 = 3x^2 \Rightarrow 9 = x^2 \Rightarrow x = \pm 3$. Since we are considering the interval [0, 4], we need only consider the critical value $x = 3$. Since the interval is a closed interval, find $f(0) = 10$, $f(3) = 64$, $f(4) = 54$. Thus, there is an absolute minimum of 10 at $x = 0$ and an absolute maximum of 64 at $x = 3$.

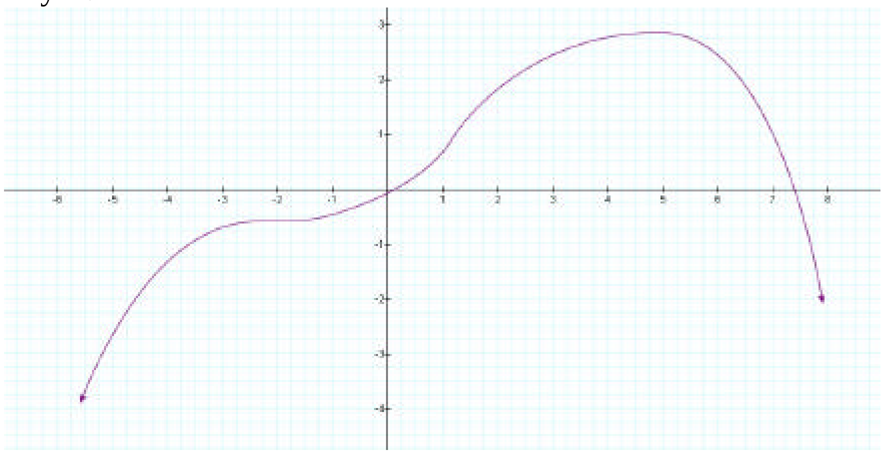
11. (6 pts.) Sketch the graph of a function that satisfies the given conditions:

$f'(-2) = f'(5) = f''(-2) = f''(1) = 0$

$f'(x) > 0$ when $x < 5$ $f'(x) < 0$ when $x > 5$

$f''(x) > 0$ when $-2 < x < 1$ $f''(x) < 0$ when $x < -2$ and $x > 1$.

ANS: One possibility is:



This function has the following properties:

horizontal tangent lines at $x = -2$ and $x = 5$ $\{ f'(-2) = f'(5) = 0 \}$,

is concave up on [–2, 1] $\{ f''(x) > 0$ when $-2 < x < 1\}$,

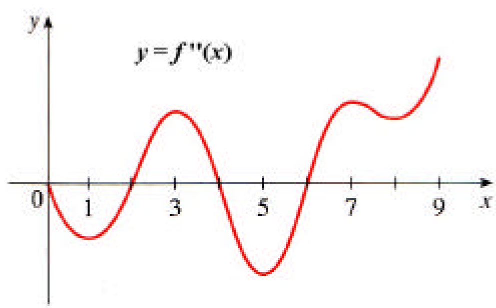
is concave down elsewhere $\{ f''(x) < 0$ when $x < -2$ and $x > 1\}$,

is increasing on (–∞, 5) $\{ f'(x) > 0$ when $x < 5 \}$,

is decreasing elsewhere $\{ f'(x) < 0$ when $x > 5 \}$,

changes concavity at $x = -2$ and $x = 1$ $\{ f''(-2) = f''(1) = 0 \}$.

12. (6 pts.) Given the graph of $f''(x)$ below, find the intervals of concavity and the inflection points for $f(x)$. EXPLAIN.



ANS: $f(x)$ is concave down when $f''(x) < 0$ and concave up when $f''(x) > 0$. Since the graph is $f''(x)$, $f''(x) < 0$ when the graph is below the x-axis and $f''(x) > 0$ when the graph is above the x-axis. Thus, $f(x)$ is concave up on (2, 4), (6, 9) and concave down on (0, 2), (4, 6). The inflection points are where $f(x)$ changes concavity or when $f''(x) = 0$. Thus the inflection points are at $x = 2, 4, 6$.

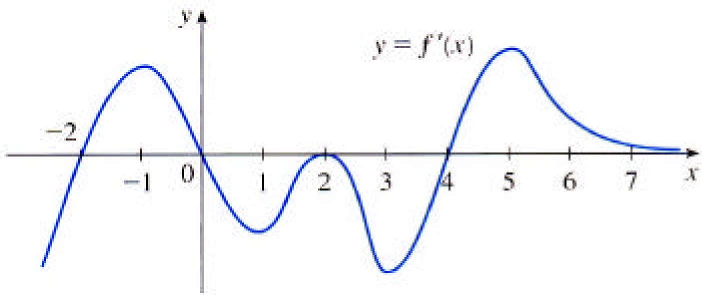
13. (6 pts.) Given the graph of $f'(x)$ below, find

a. the intervals of increase or decrease of $f(x)$

b. the relative (local) maximum and minimum values of $f(x)$

c. the intervals of concavity and the inflection points for $f(x)$

EXPLAIN.



ANS: a. $f(x)$ is decreasing when $f'(x) < 0$ and increasing when $f'(x) > 0$. Since the graph is $f'(x)$, $f'(x) < 0$ when the graph is below the x-axis and $f'(x) > 0$ when the graph is above the x-axis. Thus, $f(x)$ is increasing on $(-2, 0)$, $(4, \infty)$ and decreasing on $(-\infty, -2)$, $(0, 4)$.

b. The possible relative max/min points occur when $f''(x) = 0$. These points are at $x = -2, 0, 2, 4$. Now use the First Derivative Test to determine whether a max or min occurs at these points. If a function increases to a point ($f'(x) > 0$ or $f'(x)$ is above the x-axis) then decreases after the point ($f'(x) < 0$ or $f'(x)$ is below the x-axis), then a max occurs at the point. There is a similar argument for determining a min. Thus, a relative max occurs at $x = 0$ and a relative min occurs at $x = -2, 4$.

c. The inflection points occur when $f''(x) = 0$. Since the graph is $f'(x)$, the points where the $f''(x) = 0$ are the points where $f'(x)$ has horizontal tangent lines (i.e., where the derivative of $f'(x)$ is zero). Thus, the inflection points occur at x = $-1, 1, 2, 3, 5$. Lastly, $f(x)$ is concave down when $f''(x) < 0$ and concave up when $f''(x) > 0$. Since the graph is $f'(x)$, the second derivative is the derivative of $f'(x)$ (i.e., the second derivative describes whether $f'(x)$ is increasing ($f''(x) > 0$) or decreasing ($f''(x) < 0$). Thus, $f(x)$ is concave up ($f''(x) > 0$ or $f'(x)$ increasing up to right) on $(-\infty, -1)$, $(1, 2)$, $(3, 5)$ and is concave down ($f''(x) < 0$ or $f'(x)$ decreasing down to right) on $(-1, 1)$, $(2, 3)$, $(5, \infty)$.

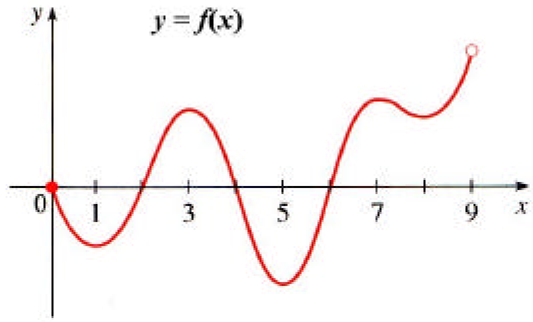
14. (6 pts.) Given the graph of $f(x)$ below, find

a. the intervals of increase or decrease $f(x)$

b. the relative (local) maximum and minimum values $f(x)$

c. the intervals of concavity and the inflection points for $f(x)$

EXPLAIN.



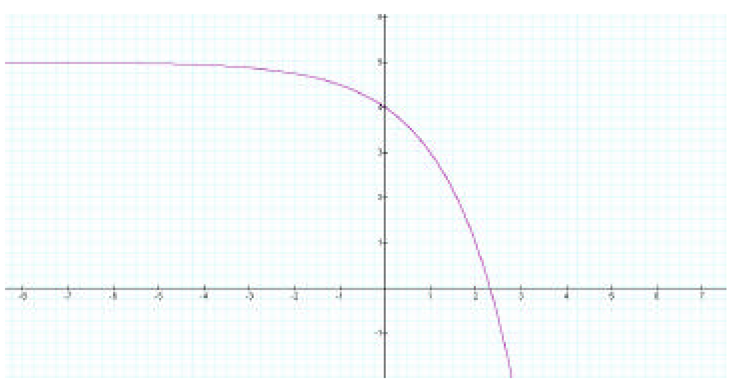
ANS: a. Since the graph is $f'(x)$, consider the graph directly. Thus, $f(x)$ is increasing on (1, 3), (5, 7) , (8, 9) and decreasing on (0, 1), (3, 5) , (7, 8).

b. A relative max occurs at $x = 0, 3, 7$ and a relative min occurs at $x = 1, 5, 8$.

c. $f(x)$ is concave up on (0, 2), (4, 6), (7.5, 9) and is concave down on (2, 4), (6, 7.5). The inflection points occur when the concavity changes and occur at x = 2, 4, 6, 7.5.

15. (5 pts.) Sketch the graph of a function whose first and second derivatives are always negative. EXPLAIN.

ANS: When the first derivative is negative, the function decreasing and when the second derivative is negative, the function concave down. Here is one possibility:



16. (6 pts.) Find the relative (local) and absolute (global) maximum and minimum value(s) and the inflection point(s) of the function, if any. EXPLAIN. [Find exact values – estimated values will not receive credit.] $f(x) = x^3 - 3x^2 - 5x + 19$

ANS: The possible max/min of a function occur when $f'(x) = 0$. $f'(x) = 3x^2 - 6x - 5 \overset{set}{=} 0$. So, by the quadratic equation, $x = \dfrac{3 \pm 2\sqrt{6}}{3}$. The possible inflection points of a function occur when $f''(x) = 0$. $f''(x) = 6x - 6 \overset{set}{=} 0$. So, the only inflection point occurs at $x = 1$. Since the domain of the function is $(-\infty, \infty)$ and since the function is odd, there are no absolute max/min. Thus, we need to check to see if a local max or min occurs at $x = \dfrac{3 \pm 2\sqrt{6}}{3}$, $x = -0.6, 2.6$. So, $f''(-0.6) < 0$, so there is a local max at $x = \dfrac{3 - 2\sqrt{6}}{3}$ and $f''(2.6) > 0$, so there is a local min at $x = \dfrac{3 + 2\sqrt{6}}{3}$.