MAT 254 – Fall Quarter 2002
Final Exam – Answers

NAME____

Show work and write clearly. Answers without work to support them will not receive full credit. Answers without correct notation will not receive full credit.

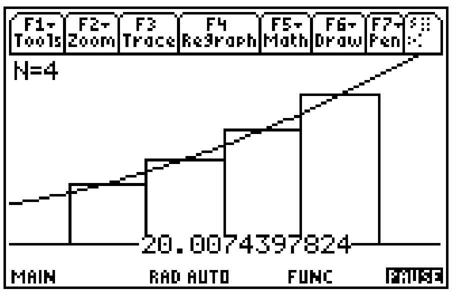
Answer *only* 11 of the 12 questions. Where necessary, estimate to 4 decimal places.

1. Without using the *allsums* program, estimate the area under the graph of $f(x) = x^2 + \frac{2}{x}$ from $x = 2$ to $x = 4$ using four approximating rectangles and midpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate? Explain.

ANS: $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}$; $x_0 = 2; x_1 = x_0 + \Delta x = 2.5; x_2 = 3; x_3 = 3.5; x_4 = 4$.

The midpoints are: $x_1^* = \frac{x_0 + x_1}{2} = 2.25; x_2^* = 2.75; x_3^* = 3.25; x_4^* = 3.75$.

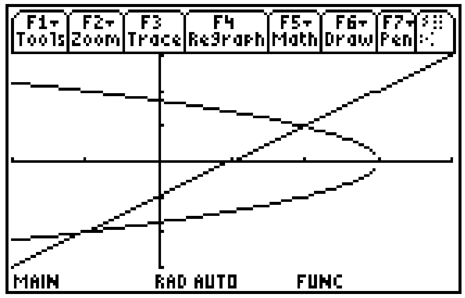
$M_3 = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$
$= f(2.25)\Delta x + f(2.75)\Delta x + f(3.25)\Delta x + f(3.75)\Delta x$
$= (5.9514)(0.5) + (8.2898)(0.5) + (11.1779)(0.5) + (14.5958)(0.5) = 20.0075.$



Since the function is concave up on [2, 4], the midpoint sum is an underestimate.

2. Sketch the region enclosed by $y = x - 1$ and $x = 3 - y^2$. Find the area.

ANS:



To graph, solve the second equation for y: $y = \pm\sqrt{3 - x}$. Horizontal rectangles are needed, so we need to integrate with respect to *y*. We need the functions in terms of *y*, i.e., *x* = *f*(*y*). The intersection points are found using the intersection feature of your calculator or by setting the functions equal to one another. Finally, the 'upper' function is the function to the right. So,

$$A = \int_{-2}^{1} \left[(3 - y^2) - (y + 1)\right] dy = \int_{-2}^{1} \left[3 - y^2 - y - 1\right] dy = \int_{-2}^{1} \left[2 - y^2 - y\right] dy$$

$$= \left[2y - \frac{y^3}{3} - \frac{y^2}{2}\right]_{-2}^{1} = \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right) = \frac{9}{2}.$$

3. a. Find the average value of $f(x) = \frac{3x}{\sqrt{1-x^2}}$ from $x = 0$ to $x = \frac{1}{2}$.

b. Find c such that average value of f equals $f(c)$. Explain.

c. Sketch the graph of the function and a rectangle whose area is the same as the area under the graph of f.

ANS:

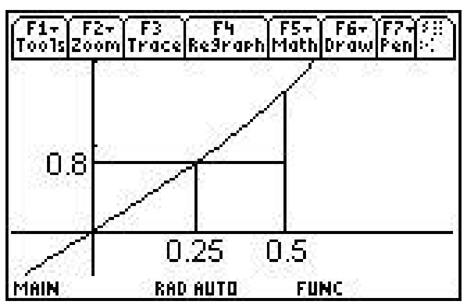
a. $f_{avg} = \frac{1}{1/2 - 0}\int_0^{1/2} \frac{3x}{\sqrt{1-x^2}}\,dx$. Use substitution, let $u = 1 - x^2$, $du = -2xdx$. So,

$$f_{avg} = -\frac{3(2)}{2}\int_1^{3/4} u^{-1/2}du = -3(2)\left[u^{1/2}\right]_1^{3/4} = -6\left[\left(\frac{3}{4}\right)^{1/2} - 1\right] = -3\sqrt{3} + 6 \cong 0.8038.$$

b. $f(c) \cong 0.8038 \Rightarrow \frac{3c}{\sqrt{1-c^2}} = 0.8038$. The easiest way to estimate c, is to graph

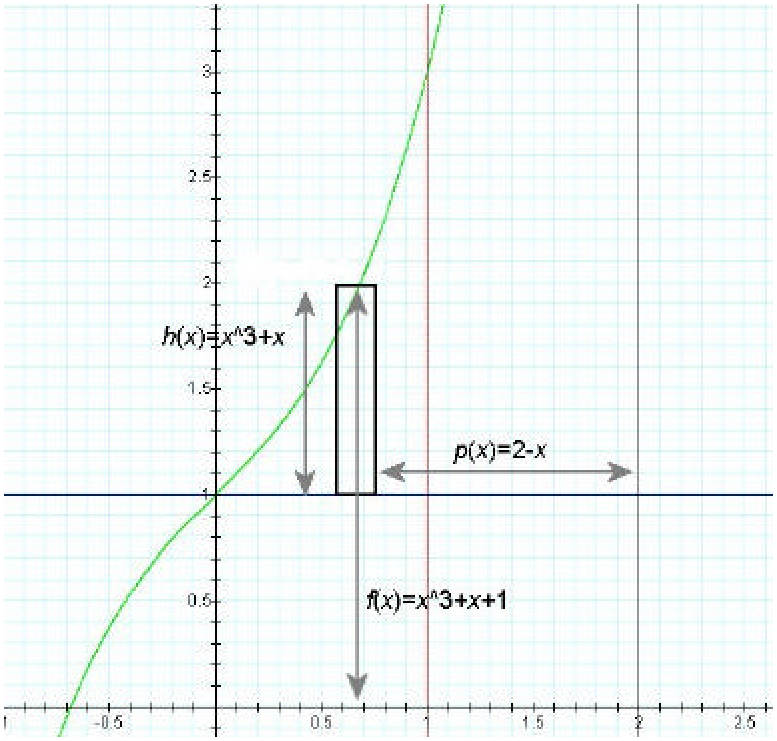
$y1 = \frac{3x}{\sqrt{1-x^2}}$ and $y2 = 0.8038$ on your calculator and find intersection: $c = 0.2588$.

c.



4. Find the volume of the solid formed by revolving the region bounded by $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 2$. Sketch the area.

ANS:



Use shell method: $h(x) = (x^3 + x + 1) - 1 = x^3 + x$ and $p(x) = 2 - x$ (i.e., axis of rotation – x).

$$V = 2p\int_0^1 p(x)h(x)dx = 2p\int_0^1 (x^3 + x)(2 - x)dx$$

$$= 2p\int_0^1 (-x^4 + 2x^3 - x^2 + 2x)dx = 2p\left[-\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} + x^2\right]_0^1$$

$$= 2p\left[-\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 1 - 0\right] = \frac{29}{30}p \ .$$

5. Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \frac{p}{4}$.

ANS: According the chain rule: $y' = \frac{1}{\cos x}(-\sin x) = -\tan x$.

Arc length = $\int_0^{p/4}\sqrt{1 + (-\tan x)^2}\,dx = \int_0^{p/4}\sqrt{1 + \tan^2 x}\,dx = \int_0^{p/4}\sqrt{\sec^2 x}\,dx = \int_0^{p/4}\sec x\,dx$.

You will need a table of integrals to find the antiderivative, so estimate using simpson with $n = 30$: $S_{30} = 0.8814$.

6. Find the area of the surface of revolution obtained by rotating the curve $y = \frac{x^5}{10} + \frac{1}{6x^3}$,

$1 \le x \le 2$ about the x-axis.

ANS:

$y' = \frac{x^4}{2} - \frac{1}{2x^4}$. So, $1 + (y')^2 = 1 + \left(\frac{x^4}{2} - \frac{1}{2x^4}\right)^2 = 1 + \frac{x^8}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4x^8} = \frac{x^8}{4} + \frac{1}{2} + \frac{1}{4x^8}$

$= \left(\frac{x^4}{2} + \frac{1}{2x^4}\right)^2$.

$$SA = 2p\int_1^2 y\sqrt{1 + (y')^2}\,dx = 2p\int_1^2 \left(\frac{x^5}{10} + \frac{1}{6x^3}\right)\sqrt{\left(\frac{x^4}{2} + \frac{1}{2x^4}\right)^2}\,dx = 2p\int_1^2 \left(\frac{x^5}{10} + \frac{1}{6x^3}\right)\left(\frac{x^4}{2} + \frac{1}{2x^4}\right)dx$$

$$= 2p\int_1^2 \left(\frac{x^9}{20} + \frac{x}{20} + \frac{x}{12} + \frac{1}{12x^7}\right)dx = 2p\int_1^2 \left(\frac{x^9}{20} + \frac{2x}{15} + \frac{1}{12x^7}\right)dx = 2p\left[\frac{x^{10}}{200} + \frac{x^2}{15} + \frac{1}{-72x^6}\right]_1^2$$

$$= 2p\left[\left(\frac{2^{10}}{200} + \frac{2^2}{15} + \frac{1}{-72(2)^6}\right) - \left(\frac{1^{10}}{200} + \frac{1^2}{15} + \frac{1}{-72}\right)\right]$$

$$= 2p\left[(5.3865) - (0.5528)\right] \cong 9.6674p \cong 30.371$$

7. Solve the initial value problem: $e^{-x^2}(y^2 - 1)y' + xy = 0$, $y(0) = 1$.

ANS: First separate the variables: $e^{-x^2}(y^2 - 1)\frac{dy}{dx} = -xy \Rightarrow \frac{(y^2 - 1)}{y}dy = -xe^{x^2}dx$. Now integrate both sides: $\Rightarrow \int \frac{(y^2 - 1)}{y}dy = \int -xe^{x^2}dx \Rightarrow \int\left(y - \frac{1}{y}\right)dy = -\frac{1}{2}\int e^u du$

$\Rightarrow \frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^u + C \Rightarrow y^2 - 2\ln|y| = -e^{x^2} + C \Rightarrow y^2 = -e^{x^2} + 2\ln|y| + C$

$\Rightarrow y = \sqrt{-e^{x^2} + 2\ln|y| + C}$. $y(0) = 1$ means when $x = 0$, $y = 1$; we use this condition to solve for C: $\Rightarrow 1 = \sqrt{-e^0 + 2\ln|1| + C} \Rightarrow 1 = \sqrt{-1 + C} \Rightarrow C = 2$. Thus,

$\Rightarrow y = \sqrt{-e^{x^2} + 2\ln|y| + 2}$.

8. Derive the formula for the derivative of $\sin^{-1}(x)$. Show all steps and be specific.

ANS: $y = \sin^{-1} x \Rightarrow x = \sin y$. The range is $\left[-\frac{p}{2}, \frac{p}{2}\right]$. Differentiate implicitly:

$1 = \cos y \cdot y'$. Solve for y′: $y' = \frac{1}{\cos y}$. Now $\cos y \geq 0$ since y ∈ $\left[-\frac{p}{2}, \frac{p}{2}\right]$. We also know, $\cos y = \sqrt{1 - \sin^2 y}$ which implies $\cos y = \sqrt{1 - x^2}$ because $x = \sin y$. Thus, $y' = \frac{1}{\sqrt{1 - x^2}}$.

9. Find the derivatives of the following:

a. $y = \sin^{-1}(x) + x\sqrt{1 - x^2}$

ANS: $y' = \frac{1}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} + x\left(\frac{1}{2}\right)\frac{1}{\sqrt{1 - x^2}}(-2x)$ from #8, product and chain rules.

Simplifying, we get $y' = \frac{1 - x^2}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} = \sqrt{1 - x^2} + \sqrt{1 - x^2} = 2\sqrt{1 - x^2}$.

b. $y = (\sin x)^{\tan x}$

ANS: Use logarithmic differentiation: $\ln y = \tan x \cdot \ln(\sin x)$ and implicitly differentiate:

$\frac{1}{y} y' = \tan x \cdot \frac{1}{\sin x} \cos x + \sec^2 x \cdot \ln(\sin x) \Rightarrow \frac{1}{y} y' = 1 + \sec^2 x \cdot \ln(\sin x)$

$\Rightarrow y' = \left(1 + \sec^2 x \cdot \ln(\sin x)\right)(\sin x)^{\tan x}$

10. Find the following limits:

a. $\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$ **ANS:** $\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$ [type $\frac{\infty}{\infty}$] $\overset{H}{=} \lim_{x \to -\infty} \frac{2x}{-e^{-x}}$ [type $\frac{\infty}{\infty}$] $\overset{H}{=} \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0$.

b. $\lim_{x \to \infty} e^{-x}\sqrt{x}$ **ANS:** [type $0 \cdot \infty$] $= \lim_{x \to \infty} \frac{\sqrt{x}}{e^x}$ [type $\frac{\infty}{\infty}$] $\overset{H}{=} \lim_{x \to \infty} \frac{1}{2\sqrt{x}e^x} = 0$.

c. $\lim_{x \to \infty}\left(x - \sqrt{x^2 + x}\right)$ **ANS:** rationalize numerator: $\lim_{x \to \infty}\left(x - \sqrt{x^2 + x}\right) \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}}$

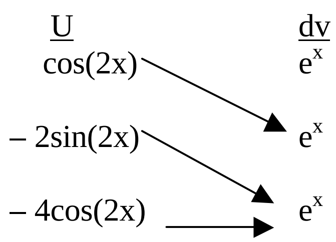
$= \lim_{x \to \infty} \frac{x^2 - \left(x^2 + x\right)}{x + \sqrt{x^2 + x}} = \lim_{x \to \infty} \frac{-x}{x + \sqrt{x^2 + x}}$. Divide all terms by the highest power of x in denominator (i.e., x): $= \lim_{x \to \infty} \frac{-x/x}{x/x + \sqrt{x^2/x^2 + x/x^2}}$. (Remember when dividing inside radical, you need to square the x). $= \lim_{x \to \infty} \frac{-1}{1 + \sqrt{1 + 1/x}} = \frac{-1}{2}$.

11. Find $\int \frac{\tan^3 x}{\cos^4 x}\,dx$

ANS: $\int \frac{\tan^3 x}{\cos^4 x}\,dx = \int \tan^3 x \sec^4 x\,dx$. Let $u = \tan x$; $du = \sec^2 x\,dx$. Now, we have $\int u^3 (\sec^2 x) du$. Use the trig identity, $\sec^2 x = 1 + \tan^2 x$. So, $\int u^3 (1 + \tan^2 x) du$ and $\tan x = u$, so $\int u^3 (1 + u^2) du = \int (u^3 + u^5) du = \frac{1}{4}u^4 + \frac{1}{6}u^6 + C = \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C$.

12. Find $\int e^x \cos(2x)dx$

ANS: Use integration by parts:



$\int e^x \cos(2x)dx = e^x \cos(2x) + 2e^x \sin(2x) - 4\int e^x \cos(2x)dx$

Add $4\int e^x \cos(2x)dx$ to both sides:

$5\int e^x \cos(2x)dx = e^x \cos(2x) + 2e^x \sin(2x)$ and

$\int e^x \cos(2x)dx = \frac{1}{5}e^x \cos(2x) + \frac{2}{5}e^x \sin(2x) + C$.