MAT 254 – Fall Quarter 2002
Test 2 - Answers

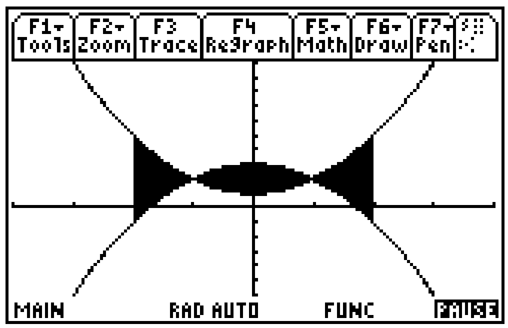
NAME____

Show work and write clearly.

1. (20 pts.) Sketch the region enclosed by the given curves. Sketch the area.

$f(x) = x^2 + 1$ and $g(x) = 3 - x^2$ between $x = -2$ and $x = 2$.

ANS: The region is shown below:



Since neither function is greater than or equal to the other function on [–2, 2], then the area is defined as:

$$A = \int_{-2}^{2} |f(x) - g(x)|$$

The intersection points of the two functions can be found by setting $f(x) = g(x)$ and solving for x. The intersection points are x = –1, 1. So the integral can be separated into three parts:

$$A = \int_{-2}^{-1} [f(x) - g(x)]dx + \int_{-1}^{1} [g(x) - f(x)]dx + \int_{1}^{2} [f(x) - g(x)]dx$$

$$= 2\int_{-2}^{-1} [f(x) - g(x)]dx + \int_{-1}^{1} [g(x) - f(x)]dx$$

$$= 2\int_{-2}^{-1} [x^2 + 1 - (3 - x^2)]dx + \int_{-1}^{1} [3 - x^2 - (x^2 + 1)]dx$$

$$= 2\int_{-2}^{-1} [2x^2 - 2)]dx + \int_{-1}^{1} [2 - 2x^2]dx$$

$$= 2\left(\frac{2x^3}{3} - 2x\right)\Bigg|_{-2}^{-1} + \left(-\frac{2x^3}{3} + 2x\right)\Bigg|_{-1}^{1} = 8$$

2. (20 pts.) a. Find the average value of $f(x) = \dfrac{\ln x}{x} + 1$ from $x = 1$ to $x = 2$.

b. Find c such that average value of f equals $f(c)$. Explain.

c. Sketch the graph of the function and a rectangle whose area is the same as the area under the graph of f.

ANS:

a. $f_{avg} = \dfrac{1}{2-1}\int_1^2 \left(\dfrac{\ln x}{x} + 1\right)dx = \int_1^2 \dfrac{\ln x}{x}dx + \int_1^2 1dx$

Use substitution for the first integral:

let $u = \ln x$; du = dx/x. When $x = 1$, $u = 0$; when $x = 2$, $u = \ln 2$.

So, we have $f_{avg} = \int_0^{\ln 2} udu + \int_1^2 1dx = \left(\dfrac{u^2}{2}\right)\Big|_0^{\ln 2} + x\Big|_1^2 = \dfrac{(\ln 2)^2}{2} + 1$

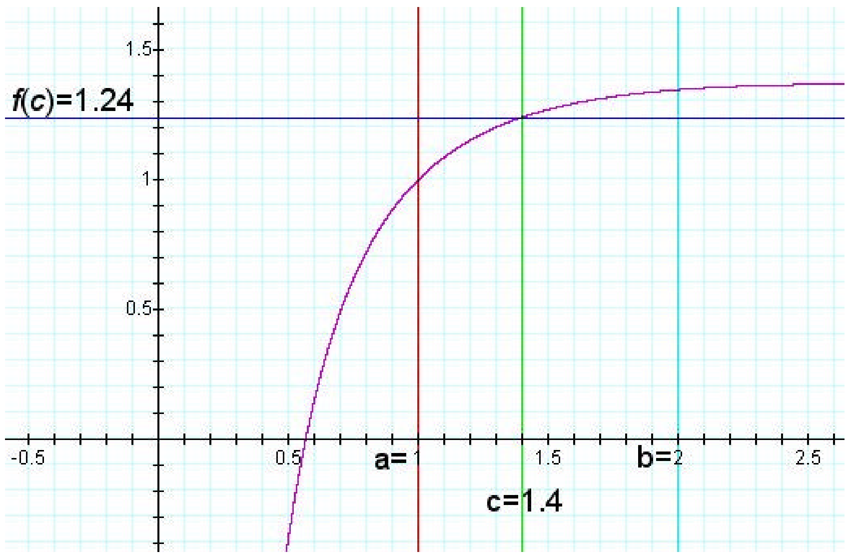
b. $f(c) = \dfrac{\ln c}{c} + 1 = \dfrac{(\ln 2)^2}{2} + 1 \approx 1.2402 \Rightarrow \dfrac{\ln c}{c} \approx 0.2402$.

There are two ways to find c:

Method 1: find root of $\ln c - 0.2402c = 0$

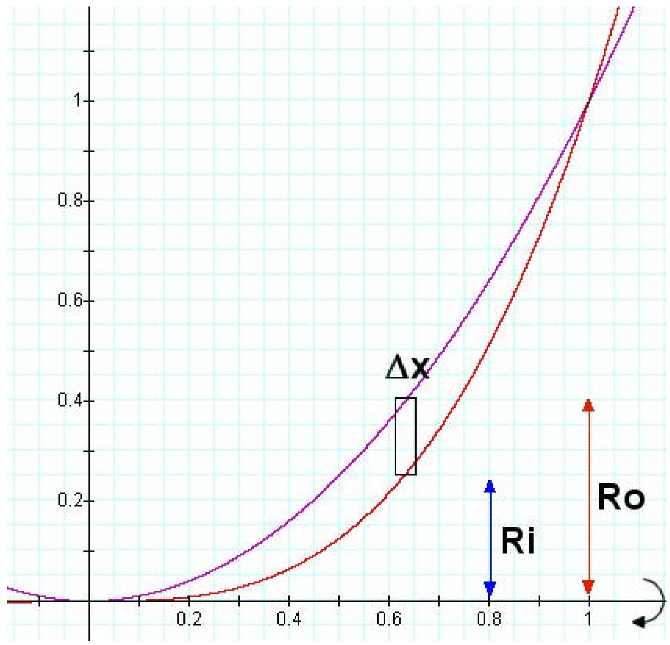
Method 2: use area of rectangle: A = (width)(height) = (2–1)(1.2402) = 1.2402. Now zoom and trace on calc until y = 1.2402. Then $c = x$ =1.4.

c.



3. (20 pts.) Use the disk method to find the volume of the solid formed by revolving the region between $y = x^2$ and $y = x^3$ about the x-axis. Sketch the area.

ANS:



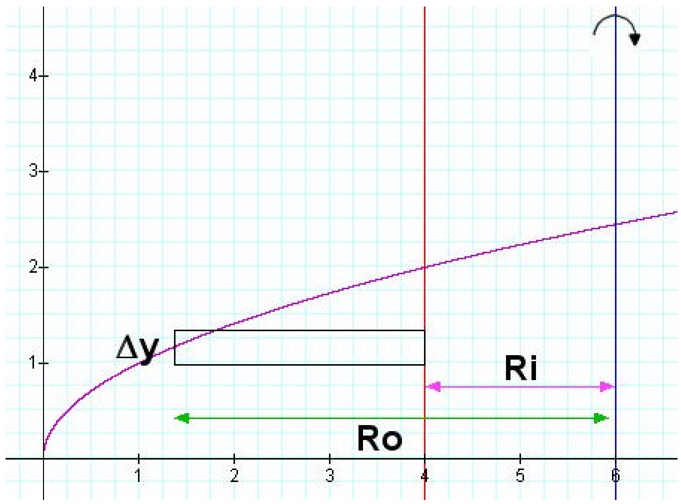
To find *a* and *b*, set functions equal to one another and solve for *x*.

$$V = p\int_a^b \left[(R_o)^2 - (R_i)^2\right]dx$$

$$V = p\int_0^1 \left[(x^2)^2 - (x^3)^2\right]dx = p\int_0^1 (x^4 - x^6)dx = p\left(\frac{x^5}{5} - \frac{x^7}{7}\right)\Big|_0^1 = \frac{2}{35}p$$

4. (20 pts.) Use the disk method to find the volume of the solid formed by revolving the region between $x = y^2$, the x-axis and $x = 4$ about the line $x = 6$. Sketch the area.

ANS:



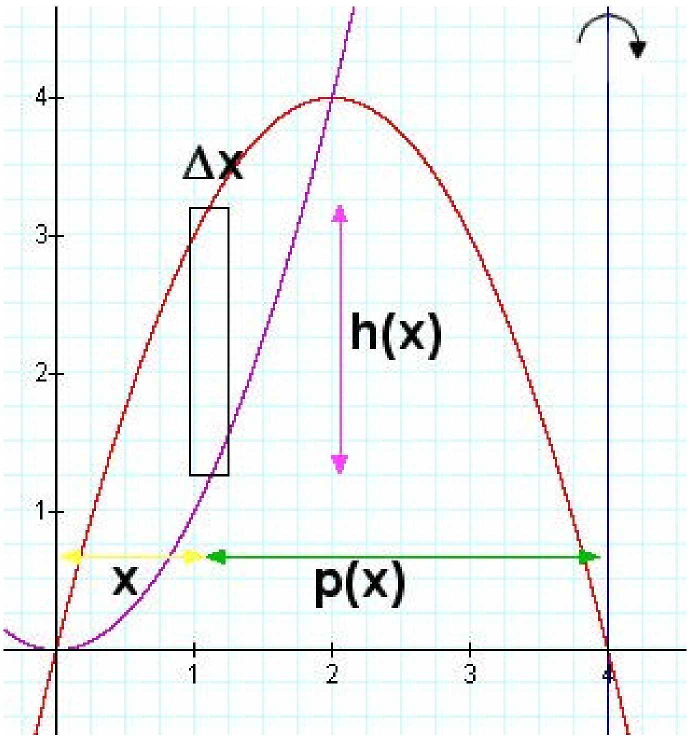
$y = 0$ at the x-axis. When $x = 4$, $y = 2$. So, $a = 0$ and $b = 2$.

$V = p\int_a^b \left[(R_o)^2 - (R_i)^2\right]dx$, where $R_i = 6 - 4 = 2$ and $R_o = 6 - x = 6 - y^2$.

$$V = p\int_0^2 \left[(6 - y^2)^2 - (2)^2\right]dy = p\int_0^2 (32 - 12y^2 + y^4)dy = p\left(32y - 4y^3 + \frac{y^5}{5}\right)\Big|_0^2 = \frac{192}{5}p$$

5. (20 pts.) Use the shell method to find the volume of the solid formed by revolving the region between $y = x^2$ and $y = 4x - x^2$ about the line $x = 4$. Sketch the area.

ANS:



$h(x)$ = top function – bottom function.
$p(x)$ = axis of rotation – x.

$$V = 2p\int_0^2 p(x)h(x)dx = 2p\int_0^2 (4-x)\left(\left(4x - x^2\right) - \left(x^2\right)\right)dx = 2p\int_0^2 (4-x)\left(4 - 2x^2\right)dx$$

$$= 2 \cdot 2p\int_0^2 (4-x)\left(2 - x^2\right)dx = 4p\int_0^2 \left(8x - 6x^2 + x^3\right)dx = 4p\left(4x^2 - 2x^3 + \frac{x^4}{4}\right)\Big|_0^2 = 16p$$