MAT 254 – Fall Quarter 2002
Test 3 - Answers

NAME____

Show work and write clearly.

1. (20 pts.) Find the length of the curve $y = \ln(\sin x)$, $\frac{\pi}{6} \le x \le \frac{\pi}{3}$.

ANS: $y' = \frac{1}{\sin x}\cos x = \cot x$.

Arc length, $s = \int_{\pi/6}^{\pi/3} \sqrt{1+\cot^2 x}\,dx = \int_{\pi/6}^{\pi/3} \sqrt{\csc^2 x}\,dx = \int_{\pi/6}^{\pi/3} \csc x\,dx$. This integral can be found using a table of integrals ($\int_a^b \csc x\,dx = \ln\left|\csc x - \cot x\right| + C$). For this test, you were permitted to use Simpson's rule with $n = 20$ to estimate the integral (0.7677).

2. (20 pts.) Find the area of the surface of revolution obtained by rotating the curve $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$ about the x-axis.

ANS: $y' = \frac{x^2}{2} - \frac{1}{2x^2}$.

Surface area, SA = $2\pi \int_1^2 f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{1+\left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\,dx$

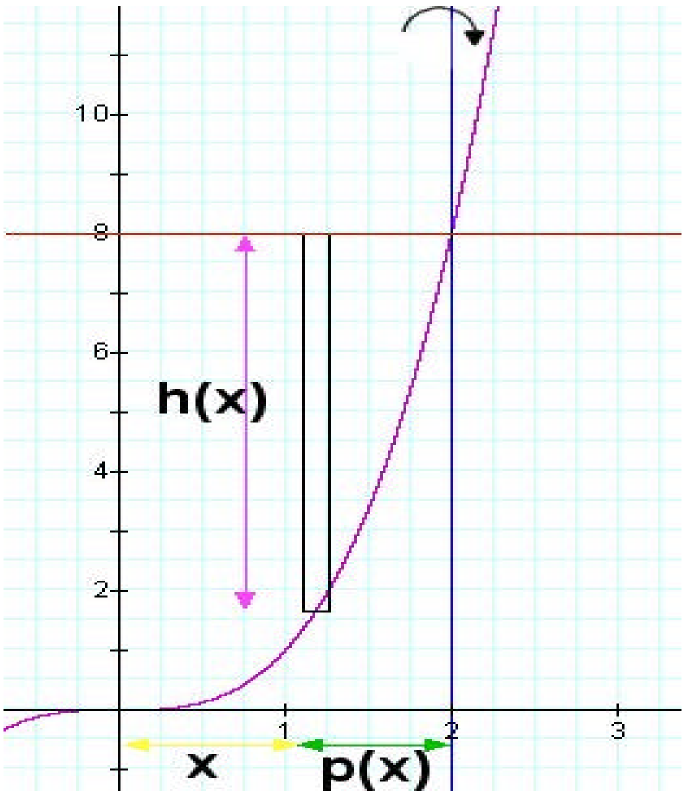
$$= 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{1+\left(\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}\right)}\,dx = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}\,dx$$

$$= 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\,dx = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx$$

$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3}\right)dx = 2\pi \left(\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right)\Big|_1^2 = \frac{47}{16}\pi$$

3. (20 pts.) Find the volume of the solid formed by revolving the region bounded by $x = \sqrt[3]{y}$ and $y = 8$, $x = 0$ about $x = 2$. Sketch the area.

ANS:



$$V = 2p\int_0^2 p(x)h(x)dx = 2p\int_0^2 (2-x)(8-x^3)dx = 2p\int_0^2 (16-8x-2x^3+x^4)dx$$

$$= 2p\left(16x - 4x^2 - \frac{x^4}{4} + \frac{x^5}{5}\right)\Big|_0^2 = \frac{144}{5}p$$

4. (20 pts.) Solve the initial value problem: $y' + 2xy = 2x^3 y$, $y(0) = 3$.

ANS: $\frac{dy}{dx} = 2x^3 y - 2xy = 2y(x^3 - x) \Rightarrow \frac{dy}{y} = 2(x^3 - x)dx \Rightarrow \int \frac{dy}{y} = 2\int (x^3 - x)dx$

$\Rightarrow \ln|y| = \frac{1}{2}x^4 - x^2 + C \Rightarrow y = e^{\frac{1}{2}x^4 - x^2 + C} = e^{\frac{1}{2}x^4 - x^2} e^C = Ce^{\frac{1}{2}x^4 - x^2}$.

When $x = 0$, $y = 3 \Rightarrow 3 = Ce^0 \Rightarrow C = 3$. Thus, $y = 3e^{\frac{1}{2}x^4 - x^2}$.

5. (20 pts.) Solve the differential equation: $y'\sqrt{x}e^{y+\sqrt{x}} = -1$.

ANS:

$\frac{dy}{dx}\sqrt{x}e^y e^{\sqrt{x}} = -1 \Rightarrow \frac{dy}{dx}e^y = \frac{-1}{\sqrt{x}e^{\sqrt{x}}} = \frac{-e^{\sqrt{x}}}{\sqrt{x}} \Rightarrow e^y dy = \frac{-e^{-\sqrt{x}}}{\sqrt{x}}dx \Rightarrow \int e^y dy = \int \frac{-e^{-\sqrt{x}}}{\sqrt{x}}dx$.

To find the integral of the left hand side use substitution: let $u = -\sqrt{x}$, $du = \frac{-1}{2\sqrt{x}}dx$. So, we have

$\int e^y dy = 2\int e^u du \Rightarrow e^y = 2e^u + C \Rightarrow e^y = 2e^u + C \Rightarrow y = \ln\left(2e^u + C\right) = \ln\left(2e^{-\sqrt{x}} + C\right)$